Chapter III
Hyperbolic sets

The existence of a transversal homoclinic point considerably complicates the orbit structure of a diffeomorphism. In order to describe this complexity, which is sometimes called deterministic chaos, we introduce the concept of a hyperbolic set and prove the shadowing lemma by means of the contraction principle. An application of the shadowing lemma shows that a transversal homoclinic point is a cluster point of homoclinic points (H. Poincaré) and a cluster point of periodic points (G. Birkhoff). In addition, the shadowing lemma allows the construction of embedded Bernoulli systems as subsystems in a neighborhood of a homoclinic orbit (S. Smale). In this way we establish orbits that are characterized by random sequences. The interpretation of such stochastic orbits will be illustrated in the simple system of the periodically perturbed mathematical pendulum.

III.1 Definition of a hyperbolic set

Hyperbolic sets are related to the dynamically unstable behavior of dynamical systems. The tangent spaces split into two invariant subspaces along which there is a contraction, respectively an expansion. The concept goes back to S. Smale and D. V. Anosov in 1967. It turns out that it is hard to get rid of hyperbolic sets by a perturbation because they are structurally stable.

Definition. The subset $\Lambda \subset \mathbb{R}^n$ is called a *hyperbolic set* of the diffeomorphism $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ if it has the following properties.

(i) Λ is compact and invariant under φ, i.e., $\Lambda = \varphi(\Lambda)$.

(ii) There exists a splitting of the tangent space in every $x \in \Lambda$,

$$\mathbb{R}^n = T_x\mathbb{R}^n = E_x^+ \oplus E_x^-, \quad x \in \Lambda,$$

which is invariant under the linearization of φ,

$$\begin{aligned} d\varphi(x)E_x^+ &= E_{\varphi(x)}^+, \\ d\varphi(x)E_x^- &= E_{\varphi(x)}^-, \end{aligned}$$

and there exist constants $c > 0$ and $0 < \vartheta < 1$ that are independent of x such that the following estimates hold true:

$$\begin{aligned} |d\varphi^j(x)\xi| &\le c\vartheta^j|\xi|, \quad \xi \in E_x^+,\ j \ge 0, \\ |d\varphi^{-j}(x)\xi| &\le c\vartheta^j|\xi|, \quad \xi \in E_x^-,\ j \ge 0. \end{aligned}$$

(iii) The splitting $E_p^+ \oplus E_p^-$ depends continuously on $p \in \Lambda$. In other words, defining the projections P_p^+ and $P_p^- = \mathbb{1} - P_p^+ \in \mathcal{L}(\mathbb{R}^n)$ onto the subspaces E_p^+ and E_p^- by

$$P_p^\pm(v_+ \oplus v_-) = v_\pm,$$

the mapping

$$\Lambda \to \mathcal{L}(\mathbb{R}^n), \quad p \mapsto P_p^\pm$$

is continuous.

Examples. If $\varphi(0) = 0$ is a hyperbolic fixed point of the diffeomorphism φ, then $\Lambda := \{0\}$ is a hyperbolic set. Every closed and invariant subset of a hyperbolic set Λ is again a hyperbolic set. In particular, the periodic orbits in Λ are hyperbolic sets.

We shall demonstrate next that the continuity of the splitting already follows from the postulated estimates in the definition of a hyperbolic set, and begin by drawing some conclusions from the estimates characterizing the subspaces $E_p^\pm$ of the tangent spaces.

Lemma III.1. $E_p^\pm = \mathrm{Im}(P_p^\pm) = \{v \in T_p\mathbb{R}^n \mid \sup_{j\geq 0} |d\varphi^{\pm j}(p)v| < \infty\}$.

Proof. The inclusion $\subset$ immediately follows from the definition. On the other hand, if $v = P_p^+ v + P_p^- v$ belongs to $T_p\mathbb{R}^n$ then, using

$$\begin{aligned} |P_p^- v| &= |d\varphi^{-j}(\varphi^j(p)) \circ d\varphi^j(p) \circ P_p^- v| \\ &\leq c\vartheta^j |d\varphi^j(p) \circ P_p^- v|, \end{aligned}$$

we obtain

$$\begin{aligned} |d\varphi^j(p)v| &= |d\varphi^j(p)P_p^- v + d\varphi^j(p)P_p^+ v| \\ &\geq |d\varphi^j(p)P_p^- v| - |d\varphi^j(p)P_p^+ v| \\ &\geq c^{-1}\vartheta^{-j}|P_p^- v| - c\vartheta^j |P_p^+ v|. \end{aligned}$$

Therefore, $\lim_{j\to\infty} |d\varphi^j(p)v| = \infty$, if $P_p^- v \neq 0$. If $\sup_{j\geq 0} |d\varphi^j(p)v| < \infty$, it also follows that $P_p^- v = 0$ and hence $v \in E_p^+$. The same argument proves that $P_p^+ v = 0$, if $\sup_{j\geq 0} |d\varphi^{-j}(p)v| < \infty$, and the lemma is proved. □

Corollary III.2. *If these estimates hold true, the sums $E_p^+ + E_p^-$ are automatically direct.*

Lemma III.3. *The projections onto E_p^+ and E_p^- are uniformly bounded, i.e., there exists a constant K such that the operator norms satisfy $\|P_p^\pm\| \leq K < \infty$ for all $p \in \Lambda$.*

Proof. We choose the integer N so large that $\lambda = c\vartheta^N < 1$ and look at the iterated map $\psi = \varphi^N$. Since Λ is a compact set there is a constant $M > 0$, such that the operator norms satisfy $\|d\psi(p)\| \leq M$ for all p in Λ. Fix p in Λ. Then, in view of the estimates in the definition of a hyperbolic set, for every $v = v_+ + v_-$ in $E_p^+ \oplus E_p^- = T_p\mathbb{R}^n$,

$$\begin{aligned} M|v| &\geq |d\psi(p)v| \\ &= |d\psi(p)v_+ + d\psi(p)v_-| \\ &\geq |d\psi(p)v_-| - |d\psi(p)v_+| \\ &\geq \lambda^{-1}|v_-| - \lambda|v_+|. \end{aligned}$$

By the triangle inequality $|v_+| = |v_+ + v_- - v_-| = |v - v_-| \leq |v| + |v_-|$ and so

$$M|v| \geq \lambda^{-1}|v_-| - \lambda(|v| + |v_-|),$$

and consequently,

$$(M + \lambda)|v| \geq (\lambda^{-1} - \lambda)|v_-|.$$

Therefore, $|P_p^- v| = |v_-| \leq K|v|$ with the constant $K = (M+\lambda)/(\lambda^{-1} - \lambda)$, and we obtain for the operator norm the estimate $\|P_p^-\| \leq K$. This holds true for every p in Λ. A similar argument shows that also $\|P_p^+\|$ is uniformly bounded and the lemma is proved. □

Lemma III.4. *The map $\Lambda \to \mathcal{L}(\mathbb{R}^n)$ defined by $p \to P_p^\pm$ is* continuous.

Proof. (1) We first show that the graph

$$\Gamma := \{(p, P_p^+) \mid p \in \Lambda\} \subset \Lambda \times \mathcal{L}(\mathbb{R}^n)$$

is a *closed* set. If p_n is a sequence in Λ satisfying $p_n \to p \in \Lambda$ and $P_{p_n}^+ \to Q^+$ in $\mathcal{L}(\mathbb{R}^n)$, we have to prove that $Q^+ = P_p^+$. Since every linear map $P_{p_n}^+$ is a projection, i.e., satisfies $(P_{p_n}^+)^2 = P_{p_n}^+$, the linear map Q^+ is also a projection. In particular,

$$\operatorname{Im} Q^+ \oplus \ker Q^+ = \mathbb{R}^n.$$

From Lemma III.3 we know that $\|P_p^+\| \leq K$ for all p, so that $\|d\varphi^j(p_n)P_{p_n}^+\| \leq cK\vartheta^j$. Hence, since $d\varphi^j(p)$ depends continuously on p, we obtain the estimate

$$\|d\varphi^j(p)Q^+\| \leq cK\vartheta^j, \quad j \geq 0.$$

Using Lemma III.1 we, therefore, conclude for the image sets of the operators that $\operatorname{Im}(Q^+) \subset \operatorname{Im}(P_p^+)$. In an analogous way, one finds $\operatorname{Im}(\mathbb{1} - Q^+) \subset \operatorname{Im}(\mathbb{1} - P_p^+)$, so that $\operatorname{Im} Q^+ = \operatorname{Im} P_p^+$ and $\ker Q^+ = \ker P_p^+$. Since Q^+ is a projection, it is thus uniquely determined and we have proved that $Q^+ = P_p^+$. Hence, Γ is a closed set.

(2) According to (1) and Lemma III.3 the set Γ is closed and bounded, hence *compact*. The projection $\pi_1\colon \Gamma \to \Lambda$ onto Λ defined by $(p, P_p^+) \mapsto p$ is bijective and continuous. Because Γ is a compact set, the inverse map π_1^{-1} is also continuous. Since the projection $\pi_2\colon \Gamma \to \mathcal{L}(\mathbb{R}^n)$ defined by $(p, P_p^+) \mapsto P_p^+$ is continuous, the composition $p \mapsto P_p^+ = \pi_2 \circ \pi_1^{-1}(p)$ is continuous.

The same arguments show that also the map $p \mapsto P_p^-$ is continuous and the lemma is proved. □

The study of the orbit structure near a homoclinic orbit will be based on the following crucial observation.

Proposition III.5. *Let* 0 *be a hyperbolic fixed point of the diffeomorphism* φ *and let* $\nu \in \mathbb{R}^n$ *be a* ***transversal*** *homoclinic point to* 0*. Then, the closure of its orbit,*

$$\Lambda := \bigcup_{j \in \mathbb{Z}} \varphi^j(\nu) \cup \{0\},$$

is a hyperbolic set.

Proof. The set Λ is invariant and compact. We define the splitting by means of the *tangent spaces* of the stable invariant manifold $W_+(0)$ and the unstable invariant manifold $W_-(0)$ which do intersect along the homoclinic orbit,

$$E_x^+ := T_x W_+(0), \quad E_x^- := T_x W_-(0).$$

Due to the transversality of the homoclinic point ν the tangent spaces split,

$$\mathbb{R}^n = T_x \mathbb{R}^n = E_x^+ \oplus E_x^-, \quad x \in \Lambda.$$

If $v \in T_p W_\pm(0)$, there exists a curve $x\colon I \to W_\pm(0)$ defined on an interval and satisfying $x(0) = p$ and $\dot{x}(0) = v$. Since the manifolds are invariant under the diffeomorphism φ the image curve $t \mapsto \varphi(x(t)) \in \mathbb{R}^n$ satisfies

$$\varphi(x(t)) \in W_\pm \quad \text{and} \quad \varphi(x(0)) = \varphi(p).$$

From

$$\frac{d}{dt}\varphi(x(t)) = d\varphi(x(t))\dot{x}(t)$$

it follows for $t = 0$ that

$$d\varphi(p)v \in T_{\varphi(p)} W_\pm(0).$$

In short, from the invariance $W_\pm = \varphi(W_\pm)$ one concludes

$$d\varphi(x) E_x^\pm = E_{\varphi(x)}^\pm.$$

The required *estimates* will be deduced from the estimates of the linearized map $d\varphi(0)$ at the hyperbolic fixed point, using that $\varphi^j(\nu) \to 0$ as $|j| \to \infty$. For this

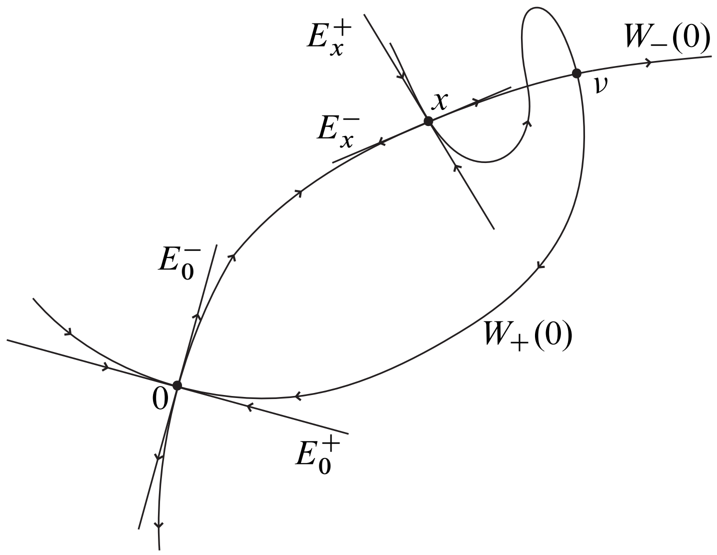


Figure III.1. Definition of the splittings $E_x^+ \oplus E_x^-$ by the tangent spaces.

purpose we introduce new coordinates near 0, for which $W_{\text{loc}}^+ = E_+$ and $W_{\text{loc}}^- = E_-$ holds true. Using the *local* representation in Theorem II.7 of the invariant manifolds (in the neighborhood Q of the fixed point) as graphs of functions,

$$y = h_+(x), \quad (x, y) \in W_{\text{loc}}^+(Q),$$
$$x = h_-(y), \quad (x, y) \in W_{\text{loc}}^-(Q),$$

we define the *coordinate transformation* $\psi(x, y) = (\xi, \eta)$ by

$$\xi = x - h_-(y),$$
$$\eta = y - h_+(x).$$

Then,

$$\psi(0) = 0, \quad d\psi(0) = \mathbb{1}$$

in view of $h_-(0) = 0$, $h_+(0) = 0$, $h'_-(0) = 0$ and $h'_+(0) = 0$. Hence, ψ is a *local diffeomorphism* near 0 by the inverse function theorem. In the new coordinates, the mapping φ is represented by

$$\hat{\varphi} = \psi \circ \varphi \circ \psi^{-1} = (\xi_1, \eta_1),$$

$$\xi_1 = f(\xi, \eta),$$
$$\eta_1 = g(\xi, \eta),$$

and the local invariant manifolds are the sets

$$W_{\text{loc}}^+(Q') = \{(\xi, 0)\} = E_+,$$
$$W_{\text{loc}}^-(Q') = \{(0, \eta)\} = E_-.$$

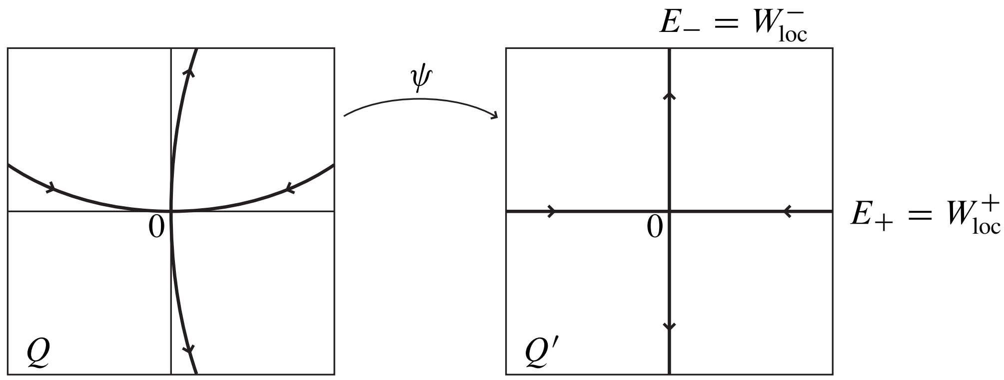


Figure III.2. The coordinate transformation in the proof of Proposition III.5.

From the $\pm$-invariance of these manifolds we deduce

$$f(0,\eta) = 0, \quad g(\xi, 0) = 0,$$

for all ξ and η small enough. Therefore, near 0, using that

$$d\varphi(0) = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix},$$

the diffeomorphism $\hat{\varphi}$ is of the form

$$\xi_1 = (A_+ + O(\xi,\eta))\xi,$$
$$\eta_1 = (A_- + O(\xi,\eta))\eta.$$

In particular,

$$d\hat{\varphi}(0) = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix},$$

and along the local stable manifold $W^+_{\mathrm{loc}}(Q')$,

$$d\hat{\varphi}(\xi, 0) = \begin{pmatrix} A_+ + O(\xi) & O(\xi) \\ 0 & A_- + O(\xi) \end{pmatrix}.$$

From $\|A_+\|, \|A_-^{-1}\| \le \alpha < 1$ it follows for $v_+ = (\xi_+, 0) \in E_+$ and $|\xi|$ small that

$$|d\hat{\varphi}(\xi,0)v_+| \le \vartheta |v_+|$$

with a constant $\alpha < \vartheta < 1$.

In the original coordinates we have $\varphi = \psi^{-1} \circ \hat{\varphi} \circ \psi$ and therefore $\varphi^j(p) = \psi^{-1} \circ \hat{\varphi}^j \circ \psi(p)$. Let K be a constant, satisfying $\|d\psi\| \le K$ and $\|d\psi^{-1}\| \le K$.

For a point p in the (perhaps smaller) neighborhood Q of 0 we find in view of

$$d\varphi^j(p)v_+ = d\psi^{-1} \circ d\hat{\varphi}^j \circ d\psi(p)v_+, \quad v_+ \in T_p W^+_{\mathrm{loc}}(Q) = E^+_p,$$

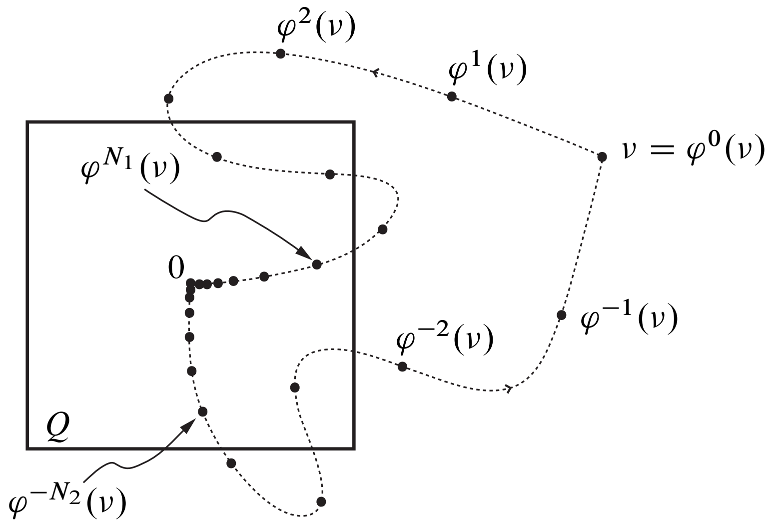


Figure III.3. The estimate in the proof of Proposition III.5.

the estimates

$$|d\varphi^j(p)v_+| \leq \vartheta^j K^2 |v_+|, \quad j \geq 0.$$

We have verified the desired estimates for the points $p \in \Lambda \cap Q$. Recalling once more the definition of a homoclinic point, we have $\varphi^j(v) \to 0$ as $|j| \to \infty$. Consequently, we find two large integers $N_1, N_2 \geq 0$ such that $\varphi^j(v) \in Q$ for all $j \geq N_1$ and $\varphi^{-j}(v) \in Q$ for all $j \geq N_2$.

Hence, introducing the integer $N = N_1 + N_2$ we conclude

$$\varphi^N(p) \in Q, \quad p \in \Lambda \setminus Q.$$

From the estimates in Q, we now obtain for all points $p \in \Lambda \setminus Q$ the estimates

$$\begin{aligned} |d\varphi^{N+n}(p)v_+| &= |d\varphi^n(\varphi^N(p)) \circ d\varphi^N(p)v_+| \\ &\leq \vartheta^n K^2 |d\varphi^N(p)v_+| \\ &\leq \max_{p\in\Lambda\setminus Q} \|d\varphi^N(p)\| \, \vartheta^n K^2 |v_+|. \end{aligned}$$

Setting $c := \max_{p\in\Lambda\setminus Q} \|d\varphi^N(p)\| \, K^2 \vartheta^{-N}$ we therefore find

$$|d\varphi^j(p)v_+| \leq c\vartheta^j |v_+|, \quad j \geq N, \ p \in \Lambda \setminus Q.$$

Since $\Lambda \setminus Q$ is a finite set of points we can choose a sufficiently large constant $C \geq c$, for which these estimates hold true also for the integers $0 \leq j < N$, and we have verified the desired estimates

$$|d\varphi^j(p)v_+| \leq C\vartheta^j |v_+|, \quad v_+ \in E_p^+, \ j \geq 0$$

for all $p \in \Lambda$ and with a constant C independent of p. The estimates for E_p^- are proved analogously and the proof of Proposition III.5 is complete. □

As in the special case of a hyperbolic fixed point, it is very convenient also for hyperbolic sets to introduce new norms in the tangent spaces with respect to which the constant c showing up in the definition of a hyperbolic set is equal to 1.

Proposition III.6 (Adapted norms). *Assume Λ to be a hyperbolic set of the diffeomorphism φ and choose a constant $\vartheta < \mu < 1$. Then, there exist equivalent norms $|\cdot|_x^*$ in $T_x\mathbb{R}^n$ for every $x \in \Lambda$, which depend* ***continuously*** *on x and satisfy*

$$
\begin{aligned}
|d\varphi(x)v_+|^*_{\varphi(x)} &\le \mu|v_+|^*_x, \quad v_+ \in E_x^+,\\
|d\varphi^{-1}(x)v_-|^*_{\varphi^{-1}(x)} &\le \mu|v_-|^*_x, \quad v_- \in E_x^-.
\end{aligned}
$$

In these norms, the linear map $d\varphi(x)$ is a contraction *in E_x^+ and an* ***expansion*** *in E_x^-.*

Proof. We choose N so large that $c(\vartheta/\mu)^N < 1$. If $v = v_+ + v_- \in E_x^+ \oplus E_x^- = T_x\mathbb{R}^n$ we define the new norm by

$$
|v_+|_x^* := \sum_{j=0}^{N-1} \mu^{-j}|d\varphi^j(x)v_+|.
$$

Using $\mu^{-N}|d\varphi^N(x)v_+| \le \mu^{-N}c\vartheta^N|v_+| \le |v_+|$, one obtains the desired estimate

$$
\begin{aligned}
|d\varphi(x)v_+|^*_{\varphi(x)} &= \sum_{j=0}^{N-1} \mu^{-j}|d\varphi^j(\varphi(x))d\varphi(x)v_+|\\
&= \mu\Big(\sum_{j=1}^{N-1} \mu^{-j}|d\varphi^j(x)v_+| + \mu^{-N}|d\varphi^N(x)v_+|\Big)\\
&\le \mu\Big(\sum_{j=0}^{N-1} \mu^{-j}|d\varphi^j(x)v_+|\Big)\\
&= \mu|v_+|_x^*.
\end{aligned}
$$

Since φ belongs to the class C^1, the new norms depend continuously on the point x. Analogously, we define the new norm for $v_- \in E_x^-$ using the inverse map φ^{-1} instead of φ. We introduce on the tangent space $T_x\mathbb{R}^n$ the max-norm

$$
|v|_x^* := \max\{|v_+|_x^*, |v_-|_x^*\},
$$

and the theorem is proved. □

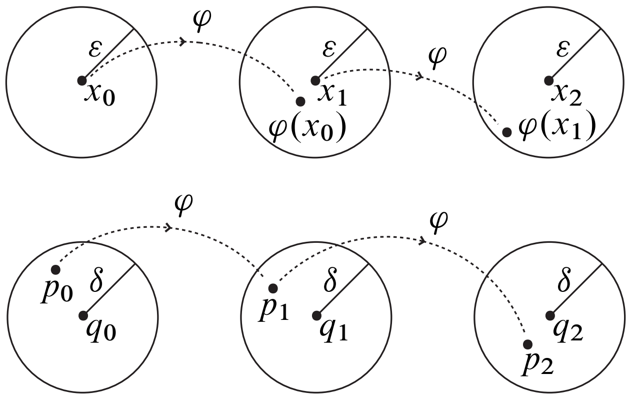


Figure III.4. An ε-pseudo orbit x (above) and a δ shadow orbit p for a sequence q (below).

III.2 The shadowing lemma

In the following, the shadowing lemma will be our main tool. If only an approximate orbit on a hyperbolic set is known, the shadowing lemma guarantees a real orbit nearby which shadows the approximate orbit. This way we shall construct orbits which are determined by their prescribed long-time behavior and not by their initial conditions. To formulate the shadowing lemma, we need some definitions.

Definition. Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism.

(i) The *sequence* $(x_j)_{j\in\mathbb{Z}}$ in $\mathbb{R}^n$ is an *orbit* of φ, if $x_{j+1} = \varphi(x_j)$ for $j \in \mathbb{Z}$.

(ii) For a given real number $\varepsilon > 0$, the sequence $(x_j)_{j\in\mathbb{Z}}$ is called an *ε-pseudo orbit* of φ, if $|x_{j+1} - \varphi(x_j)| \le \varepsilon$ for all $j \in \mathbb{Z}$.

(iii) If $\delta > 0$ and $q = (q_j)_{j\in\mathbb{Z}}$ is a sequence in $\mathbb{R}^n$, then a *δ-shadowing orbit of q* is an orbit $p = (p_j)_{j\in\mathbb{Z}}$, satisfying $|p_j - q_j| \le \delta$ for all $j \in \mathbb{Z}$.

The following theorem goes back to D. Anosov.

Theorem III.7 (Shadowing lemma). *Let Λ be a hyperbolic set of the diffeomorphism φ. Then, there exists a constant $\delta_0 > 0$ such that for every $0 < \delta \le \delta_0$ there exists an $\varepsilon = \varepsilon(\delta) > 0$ having the following property.*

*For **every** ε-pseudo orbit $q = (q_j)_{j\in\mathbb{Z}}$ of φ **on** the set Λ,*

$$q_j \in \Lambda, \quad |q_{j+1} - \varphi(q_i)| \le \varepsilon, \quad j \in \mathbb{Z},$$

*there exists **a unique** δ-shadowing orbit $p = (p_j)_{j\in\mathbb{Z}}$ of the pseudo, orbit q (for φ) in a neighborhood of Λ.*

Remark. (i) The bracket (for φ) can be replaced by the bracket (for ψ), if ψ is a diffeomorphism satisfying $|\varphi - \psi|_{C^1(U)} \le \mu$ on an open neighborhood U of Λ and if μ is sufficiently small.

(ii) The ε-pseudo orbit q does not have to lie on Λ, it is enough to require that the pseudo orbit $q = (q_j)_{j\in\mathbb{Z}}$ belongs to a sufficiently small neighborhood $V(\Lambda)$ of the hyperbolic set Λ.

Proof of Theorem III.7 [Contraction principle]. We make use of the adapted norms guaranteed by Proposition III.6.

(1) *Formulation of the problem.* If the ε-pseudo orbit $q = (q_j)_{j\in\mathbb{Z}} \subset \Lambda$ is given, we look for an orbit $p = (p_j)_{j\in\mathbb{Z}}$ satisfying $|p_j - q_j| \le \delta$ for all $j \in \mathbb{Z}$. For this purpose, we look for *corrections* $x = (x_j)_{j\in\mathbb{Z}}$, so that the sequence

$$p = q + x$$

is an *orbit*, hence satisfies

$$q_{j+1} + x_{j+1} = \varphi(q_j + x_j), \quad j \in \mathbb{Z}.$$

Rewriting this equation we look for a sequence $x = (x_j)_{j\in\mathbb{Z}}$ solving the equation

$$x_{j+1} - d\varphi(q_j)x_j = \varphi(q_j + x_j) - q_{j+1} - d\varphi(q_j)x_j =: f_j(x_j).$$

The right-hand side is *small*, if ε is small, and if $\|x\| = \sup_{j\in\mathbb{Z}} |x_j|$ is small. Indeed, due to $f_j(0) = \varphi(q_j) - q_{j+1}$ we have, by assumption, $|f_j(0)| \le \varepsilon$. In addition, the derivative satisfies $df_j(0) = d\varphi(q_j) - d\varphi(q_j) = 0$ and $df_j(x_j) = d\varphi(q_j + x_j) - d\varphi(q_j)$.

We shall solve the equation $x_{j+1} - d\varphi(q_j)x_j = f_j(x_j)$ by means of the contraction principle.

(2) *The linear problem.* We abbreviate $A_j := d\varphi(q_j) \in \mathcal{L}(\mathbb{R}^n)$. Given a sequence $(g_j)_{j\in\mathbb{Z}}$ in $\mathbb{R}^n$ we look for the sequence $x = (x_j)_{j\in\mathbb{Z}}$ solving

$$x_{j+1} - A_j x_j = g_{j+1}, \quad j \in \mathbb{Z}.$$

For this purpose, we introduce a *sequence space*. Setting $E_j = T_{q_j}\mathbb{R}^n = \mathbb{R}^n$, we define the *Banach space* of bounded sequences by

$$E = \{x = (x_j)_{j\in\mathbb{Z}} \mid x_j \in E_j,\ \|x\| < \infty\}$$

equipped with the norm $\|x\| = \sup_{j\in\mathbb{Z}} |x_j|$. We define the linear map $A \in \mathcal{L}(E)$ by its restrictions $A|_{E_j} := A_j : E_j \to E_{j+1}$, as

$$(A(x))_{j+1} := A_j x_j.$$

We want to solve the operator equation $(\mathbb{1} - A)x = g$ in the Banach space E.

Lemma III.8. *If $(q_j)_{j\in\mathbb{Z}}$ is an ε-pseudo orbit on Λ, and if ε is sufficiently small, then the linear map $\mathbb{1} - A \in \mathcal{L}(E)$ is a continuous isomorphism whose inverse map $L := (\mathbb{1} - A)^{-1} \in \mathcal{L}(E)$ is also continuous and has the finite norm $\|L\| < \infty$.*

Proof. We introduce the notation $E_j = E_j^+ \oplus E_j^- = P_{q_j}^+ E_j \oplus P_{q_j}^- E_j$. Given the sequence $g = (g_j) \in E$ we look for a sequence $x = (x_j) \in E$ solving the equation $x_{j+1} - d\varphi(q_j)x_j = g_{j+1}$ for $j \in \mathbb{Z}$, or

$$x_{j+1} = d\varphi(q_j)x_j + g_{j+1}, \quad j \in \mathbb{Z}.$$

With respect to the above splitting we obtain the equivalent equations

$$(*) \qquad \begin{cases} P_{q_{j+1}}^+ x_{j+1} = P_{q_{j+1}}^+ d\varphi(q_j)x_j + P_{q_{j+1}}^+ g_{j+1}, \\ P_{q_{j+1}}^- x_{j+1} = P_{q_{j+1}}^- d\varphi(q_j)x_j + P_{q_{j+1}}^- g_{j+1}. \end{cases}$$

The splitting $E_j^+ \oplus E_j^-$ is not invariant under the linearized map $d\varphi(q_j)$, since q is not an orbit. However, along the orbit we know from the definition of the hyperbolicity of the set Λ that

$$P_{\varphi(q_j)}^{\pm} d\varphi(q_j)x_j = d\varphi(q_j)P_{q_j}^{\pm} x_j, \quad j \in \mathbb{Z}.$$

Thus, the equation $(*)$ is equivalent to the following two equations of $(**)$ and $(***)$,

$$(**) \quad P_{q_{j+1}}^+ x_{j+1} = d\varphi(q_j)P_{q_j}^+ x_j + P_{q_{j+1}}^+ g_{j+1} + [P_{q_{j+1}}^+ - P_{\varphi(q_j)}^+]d\varphi(q_j)x_j,$$

$$\begin{aligned} (***) \qquad P_{q_j}^- x_j &= d\varphi(q_j)^{-1} P_{\varphi(q_j)}^- x_{j+1} - d\varphi(q_j)^{-1} P_{q_{j+1}}^- g_{j+1} \\ &\quad + d\varphi(q_j)^{-1}[P_{q_{j+1}}^- - P_{\varphi(q_j)}^-](x_{j+1} - d\varphi(q_j)x_j). \end{aligned}$$

We introduce the map

$$\Phi\colon E \to E, \quad x = (x_j) \mapsto (\Phi(x)_j), \quad \Phi(x)_j := P_{q_j}^+ \Phi(x)_j + P_{q_j}^- \Phi(x)_j,$$

where $P_{q_j}^+ \Phi(x)_j$ is defined by the right-hand side of the equation $(**)$ and $P_{q_j}^- \Phi(x)_j$ by the right-hand side of the equation $(***)$. By construction the desired sequence is a fixed point of this map,

$$\Phi(x) = x \Longleftrightarrow x_{j+1} - d\varphi(q_j)x_j = g_{j+1}.$$

Since Λ is compact,

$$\sup_{q\in\Lambda} \|d\varphi(q)\| \le K, \quad \sup_{q\in\Lambda} \|d\varphi(q)^{-1}\| \le K$$

for a constant K and the mappings $q \mapsto P_q^{\pm}\colon \Lambda \to \mathcal{L}(\mathbb{R}^n)$ are not only continuous, but *uniformly continuous*. Hence, for every given $\varepsilon' > 0$ there exists an $\varepsilon > 0$ such that

$$\|P_{q_{j+1}}^{\pm} - P_{\varphi(q_j)}^{\pm}\| \le \varepsilon' \quad \text{for all } j \in \mathbb{Z}, \quad \text{if} \quad |q_{j+1} - \varphi(q_j)| \le \varepsilon \quad \text{for all } j \in \mathbb{Z},$$

i.e., if the sequence q is an ε-pseudo orbit. Since Λ is hyperbolic, we have (in the adapted norms) the estimates

$$
\begin{aligned}
|d\varphi(q_j)P^+_{q_j}x_j| &\le \vartheta|x_j|,\\
|d\varphi(q_j)^{-1}P^-_{q_j}x_{j+1}| &\le \vartheta|x_{j+1}|,
\end{aligned}
$$

with a constant $0 \le \vartheta < 1$. Using this, we shall estimate the Lipschitz constant of the map Φ. Recalling the definition of the norms and using the notation $a \vee b := \max\{a, b\}$, we have

$$
\begin{aligned}
\|\Phi(x) - \Phi(y)\| &= \sup_{j\in\mathbb{Z}} |\Phi(x)_j - \Phi(y)_j|\\
&= \sup_{j\in\mathbb{Z}} \big[\, |P^+_{q_j}\Phi(x)_j - P^+_{q_j}\Phi(y)_j| \vee |P^-_{q_j}\Phi(x)_j - P^-_{q_j}\Phi(y)_j| \,\big].
\end{aligned}
$$

The stable part is estimated as

$$
\begin{aligned}
|P^+_{q_j}\Phi(x)_j - P^+_{q_j}\Phi(y)_j| = \big| & d\varphi(q_j)P^+_{q_j}(x_j - y_j)\\
&+ [P^+_{q_{j+1}} - P^+_{\varphi(q_j)}]d\varphi(q_j)(x_j - y_j)\big|\\
\le{}& \vartheta|x_j - y_j| + \varepsilon' K|x_j - y_j|.
\end{aligned}
$$

For the unstable part we get

$$
\begin{aligned}
|P^-_{q_j}\Phi(x)_j - P^-_{q_j}\Phi(y)_j| = \Big| & d\varphi(q_j)^{-1}P^-_{\varphi(q_j)}(x_{j+1} - y_{j+1})\\
&+ d\varphi(q_j)^{-1}[P^-_{q_{j+1}} - P^-_{\varphi(q_j)}]\\
&\cdot \big[x_{j+1} - y_{j+1} - d\varphi(q_j)(x_j - y_j)\big]\Big|\\
\le{}& \vartheta|x_{j+1} - y_{j+1}| + \varepsilon' K|x_j - y_j| + \varepsilon' K^2|x_j - y_j|.
\end{aligned}
$$

Taking the supremum over $j \in \mathbb{Z}$,

$$
\|\Phi(x) - \Phi(y)\| \le (\vartheta + \varepsilon' K + \varepsilon' K^2)\|x - y\|
$$

for all $x, y \in E$. If we choose $\varepsilon' > 0$ so small that $(\vartheta + \varepsilon' K + \varepsilon' K^2) =: \alpha^* < 1$, the map $\Phi\colon E \to E$ is a *contraction*. The unique fixed point $x = (x_j)_{j\in\mathbb{Z}} \in E$ of the map satisfies, in view of the equations $(**)$, $(***)$ and of Lemma III.3, the estimate

$$
\|x\| = \|\Phi(x)\| \le \|\Phi(x) - \Phi(0)\| + \|\Phi(0)\| \le \alpha^*\|x\| + K'\|g\|,
$$

with a constant $K' > 0$ and therefore,

$$
\|x\| \le \frac{K'}{1 - \alpha^*}\|g\|.
$$

In view of $x = (\mathbb{1} - A)^{-1}g$, we have verified the estimate

$$\|(\mathbb{1} - A)^{-1}\| \leq \frac{K'}{1 - \alpha^*},$$

and Lemma III.8 is proved. □

(3) *The nonlinear problem.* Let $r > 0$. We denote the closed balls of radius r in E_j and in E by $B_j(r) := \{x_j \in E_j \mid |x_j| \leq r\}$ and by $B(r) := \{x \in E \mid \|x\| \leq r\}$. We want to solve the equations

$$x_{j+1} - A_j x_j = f_j(x_j),$$

for a sequence $x = (x_j)_{j \in \mathbb{Z}}$ satisfying $x_j \in E_j$, while the sequence of maps $f_j : B_j(r) \subset E_j \to E_{j+1}$ is given. Introducing the mapping

$$F : B(r) \subset E \to E \quad \text{by} \quad F(x)_{j+1} = f_j(x_j),$$

our equation can be written as $(\mathbb{1} - A)x = F(x)$ or as

$$x = LF(x), \quad x \in B(r),$$

with the continuous linear map $L = (\mathbb{1} - A)^{-1}$. In the following, we write $|\cdot|$ instead of $\|\cdot\|$ for the norm on E and reserve the notation $\|\cdot\|$ for the operator norm.

Lemma III.9. *Let $F : B(r) \subset E \to E$ be a map. Assume that the real number $\alpha > 0$ is so small that $\alpha\|L\| \leq 1/2$. If $|F(0)| \leq \alpha r$ and $|F(x) - F(y)| \leq \alpha|x - y|$ for all $x, y \in B(r)$, then the equation $x = LF(x)$ has a* ***unique*** *solution $x \in B(r)$. This solution satisfies the estimate*

$$|x| \leq 2\|L\|\,|F(0)|.$$

Proof. Set $G(x) := LF(x)$. We claim that

(i) $G : B(r) \to B(r)$, and
(ii) $|G(x) - G(y)| \leq \frac{1}{2}|x - y|$ for all $x, y \in B(r)$.

In order to prove the claim we take $x, y \in B(r)$ and estimate, using the assumptions,

$$|G(x) - G(y)| \leq \|L\|\,|F(x) - F(y)| \leq \alpha\|L\|\,|x - y| \leq \frac{1}{2}|x - y|.$$

Observing that $|G(0)| = |LF(0)| \leq \|L\|\,|F(0)| \leq \|L\|\alpha r \leq r/2$, we obtain

$$|G(x)| \leq |G(x) - G(0)| + |G(0)| \leq \frac{1}{2}|x| + \frac{r}{2} \leq r,$$

and the claim is proved. Since the metric space $B(r)$ is *complete*, there exists a unique fixed point $x = G(x)$ satisfying $|x| \le r$ and due to $|x| = |G(x)| \le |G(x) - G(0)| + |G(0)| \le \frac{1}{2}|x| + |G(0)|$, we arrive at the desired estimate

$$|x| \le 2|G(0)| \le 2\|L\|\,|F(0)|.$$

This concludes the proof of Lemma III.9. □

Finally, we apply the lemma to our situation and complete the proof of the shadowing lemma. We recall that

$$|F(0)| = \sup_j |f_j(0)| = \sup_j |\varphi(q_j) - q_{j+1}| \le \varepsilon.$$

We choose α so small that $\alpha\|L\| \le \frac{1}{2}$. Since Λ is compact and $df_j(0) = 0$, we find a radius $r_0 = \delta_0$ such that $\|df_j(x_j)\| \le \alpha$ for every $x_j \in B_j(r_0)$ and all $j \in \mathbb{Z}$. By the mean value theorem we conclude $|F(x) - F(y)| \le \alpha|x - y|$ for $x, y \in B(r_0)$. If now $r \le \delta_0$ and if $\varepsilon \le \alpha r$, we conclude from Lemma III.9 that the statement of the shadowing lemma holds true with the constant $\delta = r$. This completes the proof of Theorem III.7. □

Proof of the remark following the shadowing lemma. Let $|\varphi - \psi|_{C^1(U)} \le \mu$ where U is a neighborhood of Λ. Replacing the maps $f_j(x_j)$ in the above proof by the maps

$$f_j'(x_j) = \psi(q_j - x_j) - q_{j+1} - d\psi(q_j)x_j,$$

we can argue as above, if μ is sufficiently small. As for the second part of the remark, we choose $\eta > 0$ such that the $\hat{\varepsilon}$-pseudo orbit $q = (q_j)_{j\in\mathbb{Z}}$ lies in the η-neighborhood of Λ. Choosing a sequence q' *on* Λ satisfying $|q_j - q_j'| \le \eta$ for all $j \in \mathbb{Z}$, it follows that

$$\begin{aligned} |q_{j+1}' - \varphi(q_j')| &\le |q_{j+1}' - q_{j+1}| + |q_{j+1} - \varphi(q_j)| + |\varphi(q_j) - \varphi(q_j')| \\ &\le \eta + \hat{\varepsilon} + \eta \sup_{x\in\Lambda} \|d\varphi(x)\| \\ &=: \varepsilon, \end{aligned}$$

so that q' is an ε-pseudo orbit *on* Λ. If $\eta, \hat{\varepsilon}$ are sufficiently small, we can apply the first part of the theorem to the pseudo orbit $q' \subset \Lambda$ to obtain a $(\delta + \eta)$-shadowing orbit for the pseudo orbit q. □

As a first application of the shadowing lemma, we shall prove the closing lemma of Anosov.

Theorem III.10 (Closing lemma of Anosov). *We consider the hyperbolic set Λ of the diffeomorphism φ and let ε, δ be as in the shadowing lemma. If there exist a point $x \in \Lambda$ and an integer $N \ge 1$ satisfying*

$$|\varphi^N(x) - x| \le \varepsilon,$$

then there exists a point y in a δ neighborhood $U_\delta(\Lambda)$ of Λ satisfying

$$\varphi^N(y) = y.$$

Moreover, the periodic orbit $y, \varphi(y), \dots, \varphi^N(y) = y$ *lies in a δ-neighborhood of the set* $\{x, \varphi(x), \dots, \varphi^N(x)\}$.

Proof [Uniqueness of the δ-shadowing orbit]. We define the ε-pseudo orbit $q = (q_j)_{j\in\mathbb{Z}}$ by the N-periodic continuation of the finite piece of the orbit

$$\begin{array}{ccccc} x & \varphi(x) & \varphi^2(x) & \dots & \varphi^{N-1}(x) \\ \| & \| & \| & & \| \\ q_0 & q_1 & q_2 & \dots & q_{N-1}\ , \end{array}$$

so that $q_{j+N} = q_j$ for all $j \in \mathbb{Z}$. By the shadowing lemma there exists a unique δ-shadowing orbit $p = (p_j)_{j\in\mathbb{Z}}$ of the pseudo orbit q and we claim that

$$p_{j+N} = p_j, \quad j \in \mathbb{Z}.$$

To prove the claim, we introduce the shifted orbit sequence $\hat{p} = (\hat{p}_j)_{j\in\mathbb{Z}}$ by $\hat{p}_j = p_{j+N}$. Then, also $\hat{p}$ is a δ-shadowing orbit of the pseudo orbit q, since

$$|\hat{p}_j - q_j| = |p_{j+N} - q_j| = |p_{j+N} - q_{j+N}| \le \delta$$

holds true for all $j \in \mathbb{Z}$. From the uniqueness of the δ-shadowing orbit which shadows the pseudo orbit q, we conclude that $\hat{p} = p$, so that the orbit p is indeed the desired periodic orbit, as claimed in the theorem. □

III.3 Orbit structure near a homoclinic orbit, chaos

In the following we consider a transversal homoclinic point ν at which, by definition, the stable and unstable invariant manifolds issuing form a hyperbolic fixed point of the diffeomorphism φ intersect transversally. Assuming as before the fixed point to be the origin 0 we denote by

$$\Lambda = \overline{\mathcal{O}(\nu)} = \bigcup_{j\in\mathbb{Z}} \varphi^j(\nu) \cup \{0\} = \mathcal{O}(\nu) \cup \mathcal{O}(0)$$

the closure of the homoclinic orbit which consists of two orbits. The compact set Λ is a hyperbolic set of the diffeomorphism φ in view of Proposition III.5 and so we can use the shadowing lemma in order to prove first that the homoclinic point ν is a *cluster point* of other *homoclinic points* belonging to 0 and at the same time also a cluster point of *periodic points*.

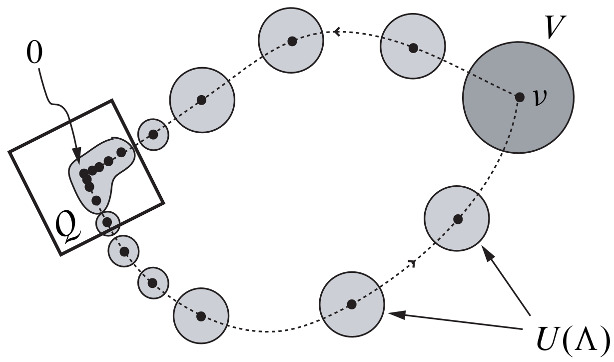


Figure III.5. Homoclinic orbit with neighborhood

Theorem III.11 (G. Birkhoff). *We assume that ν is a transversal homoclinic point belonging to the hyperbolic fixed point 0 of the diffeomorphism φ. Let V be an open neighborhood of ν and let $U = U(\Lambda)$ be an open neighborhood of $\Lambda = \overline{\mathcal{O}(\nu)}$.*

Then, there exist infinitely many periodic points in V whose orbits are contained in the open set U. More precisely, there exists an integer $N_0 = N_0(U, V)$ such that for every integer $N \geq N_0$ there exists a periodic point $p \in V$ having the minimal period N.

Proof [Shadowing lemma]. The hyperbolic set $\Lambda = \mathcal{O}(\nu) \cup \mathcal{O}(0)$ consists of two orbits of φ. If ε and δ_0 are as in the shadowing lemma we choose $0 < \delta \leq \delta_0$ so small that the δ neighborhood of ν lies in V and the δ neighborhood of Λ in the prescribed open set $U(\Lambda)$. We denote by Q the ε neighborhood of the fixed point 0. According to the definition of a homoclinic point ν there exists an integer j_0 such that

$$\varphi^j(\nu) \in Q, \quad |j| \geq j_0.$$

The crucial observation is now the following. Inside of Q it is possible to *jump* from the homoclinic orbit to the fixed point orbit of 0 and back to the homoclinic orbit by committing only an error smaller than $\leq \varepsilon$ as illustrated in Figure III.7.

We use this to construct the ε-pseudo orbits $q = (q_j)_{j\in\mathbb{Z}} \subset \Lambda$ *on* the hyperbolic set Λ having a prescribed *minimal* period, as follows.

$$\begin{array}{ccccccccccc} \nu & \varphi(\nu) & \dots & \varphi^{j_0}(\nu) & 0 & \dots & 0 & \varphi^{-j_0}(\nu) & \dots & \varphi^{-1}(\nu) \\ \| & \| & & \| & \| & & \| & \| & & \| \\ q_0 & q_1 & \dots & q_{j_0} & q_{j_0+1} & \dots & q_{j_0+k} & q_{j_0+k+1} & \dots & q_{2j_0+k} \end{array},$$

where the hyperbolic fixed point 0 is visited k-times in succession ($k \geq 1$) and where the scheme is repeated periodically. The integer k can be chosen arbitrarily and determines the minimal period of the pseudo orbit q. The sequence q is a

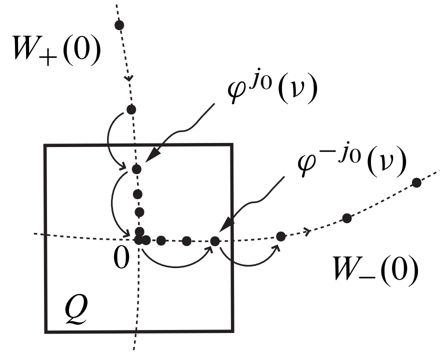


Figure III.6. Jump of the ε-pseudo orbit from $W_+(0)$ onto 0 and then into $W_-(0)$.

periodic ε-pseudo orbit on Λ, namely,

$$q_{j+N} = q_j, \quad j \in \mathbb{Z},$$

where $N = 2j_0 + k + 1$.

In view of the shadowing lemma the pseudo orbit q is shadowed by the unique δ-shadowing orbit $p = (p_j)_{j\in\mathbb{Z}}$ given by $p_j = \varphi^j(p_0)$. By construction, $p_0 \in V$ and $\mathcal{O}(p_0) \subset U(\Lambda)$. Since also the shifted sequence $\hat{p} = (\hat{p}_j)_{j\in\mathbb{Z}}$, defined by $\hat{p}_j = p_{j+N}$, is a δ-shadowing orbit of the pseudo orbit q, it follows from the *uniqueness* that $\hat{p} = p$, hence

$$p_{j+N} = p_j, \quad j \in \mathbb{Z}.$$

Therefore, the shadowing orbit p is a *periodic* orbit of φ having the minimal period N. The theorem follows if we set $N_0 = 2j_0 + 2$.

$\square$

The next result goes back to H. Poincaré. It explains why the transversal homoclinic point forces the invariant manifolds $W_+(0)$ and $W_-(0)$ issuing from the hyperbolic fixed point to double back and pile up on themselves as illustrated in Figure III.7.

Theorem III.12 (H. Poincaré). *We assume that ν is a transversal homoclinic point associated with the hyperbolic fixed point* 0 *of the diffeomorphism φ. Let V be an open neighborhood of ν and let $U = U(\Lambda)$ be an open neighborhood of $\Lambda = \overline{\mathcal{O}(\nu)}$. Then there exist infinitely many homoclinic points associated with* 0 *in V, whose orbits run in U, and which are distinguished by two* ***rotation numbers*** *$r^\pm$.*

Proof [Shadowing lemma]. We again construct a suitable ε-pseudo orbit on Λ using the same notation as in the previous proof and let $r^\pm \in \mathbb{N}_0$ be two integers. Set

$$\begin{array}{cccccccc} \nu & \varphi(\nu) & \dots & \varphi^{j_0}(\nu) & 0 & \varphi^{-j_0}(\nu) & \dots & \varphi^{-1}(\nu) \\ \| & \| & & \| & \| & \| & & \| \\ q_0 & q_1 & \dots & q_{j_0} & q_{j_0+1} & q_{j_0+2} & \dots & q_{2j_0+1}\,. \end{array}$$

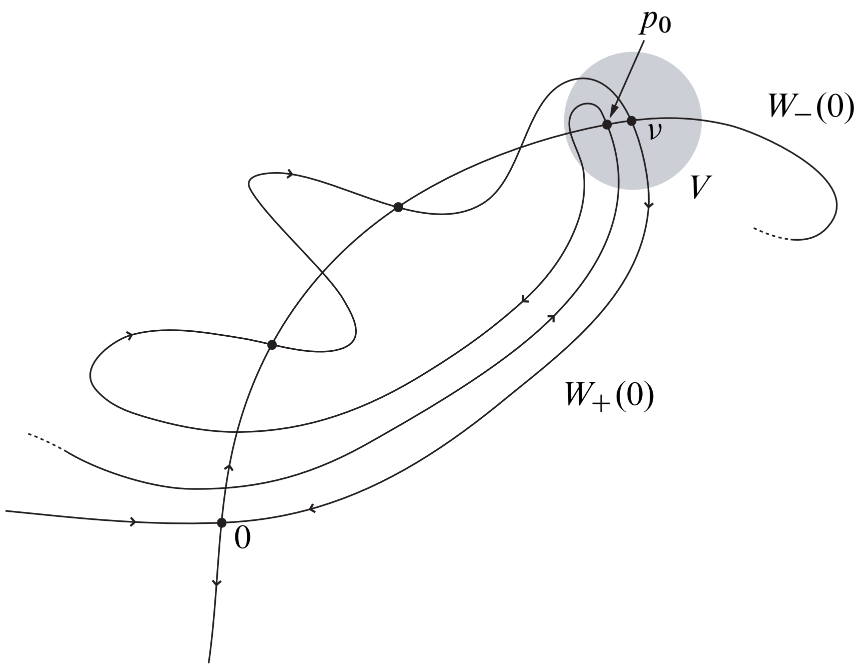


Figure III.7. One of the infinitely many homoclinic points p_0 near ν.

We repeat this finite sequence on the right r^+-times and on the left r^--times, then we add on the left $\varphi^{-j_0}(\nu), \dots, \varphi^{-1}(\nu)$ and on the right $\nu, \varphi(\nu), \dots, \varphi^{j_0}(\nu)$. Finally, we add on the left respectively on the right the infinite sequences $\dots, 0, 0, 0$ resp. $0, 0, 0, \dots$, which belong to the orbit of the fixed point 0. Hence, after choosing the integer j_0 and the open neighborhood Q as in the proof of the previous theorem, we have constructed an ε-pseudo orbit q on the hyperbolic set Λ, for which it holds true that

$$q_j = 0, \quad |j| \geq M,$$

for a suitable constant M.

This pseudo orbit is shadowed by the unique δ-shadowing orbit $p = (p_j)_{j \in \mathbb{Z}}$ which satisfies, by construction, $\mathcal{O}(p_0) \subset U$ and $p_0 \in V$ and, *in addition*,

$$|p_j| \leq \delta, \quad |j| \geq M.$$

Therefore, for $|j|$ large, all the orbit points lie in the δ neighborhood Q_δ of the hyperbolic fixed point 0. Hence, they lie on the *local manifolds* $W^+_{\mathrm{loc}}(Q_\delta)$ respectively $W^-_{\mathrm{loc}}(Q_\delta)$, introduced in II.2. Due to Theorem II.7, $\varphi^j(p_0) \to 0$ as $|j| \to \infty$, if we choose δ sufficiently small. Therefore, p_0 is a *homoclinic* point in V, and, by construction, $p_0 \neq \nu$. Shadowing orbits belonging to different rotation numbers are different from each other. The proof of the theorem is complete. $\square$

More generally, we can consider two hyperbolic fixed points $x^* \neq y^*$ of the diffeomorphism φ. If the invariant manifolds intersect *transversally* in the heteroclinic

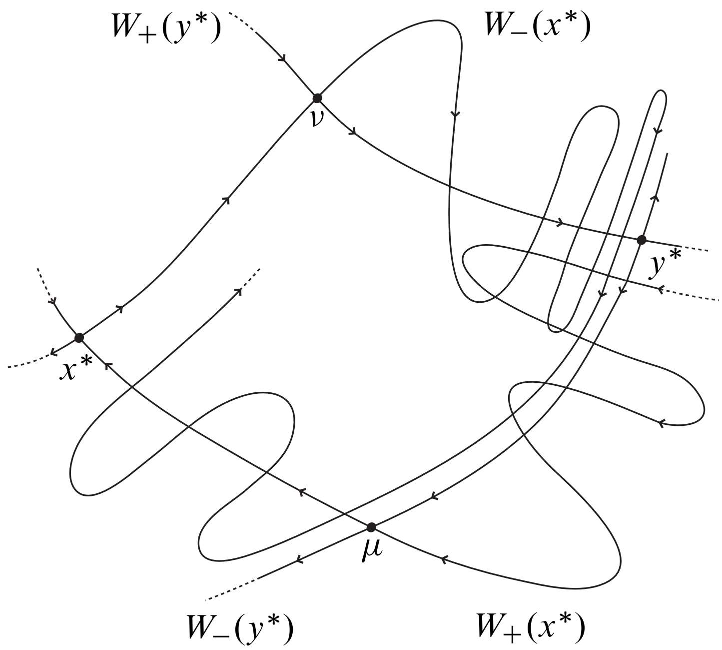


Figure III.8. Transversal heteroclinic points ν, μ belonging to the hyperbolic fixed points x^*, y^*.

points

$$\nu \in W_-(x^*) \cap W_+(y^*),$$
$$\mu \in W_+(x^*) \cap W_-(y^*),$$

as illustrated in Figure III.9, then $\varphi^j(\nu) \to y^*$ as $j \to \infty$ and $\varphi^j(\nu) \to x^*$ as $j \to -\infty$; and analogously for the intersection point μ. The set

$$\Lambda := \mathcal{O}(\nu) \cup \mathcal{O}(\mu) \cup \{x^*\} \cup \{y^*\},$$

consisting of four orbits is a *hyperbolic set*. Let $U(\Lambda)$ an open neighborhood of Λ. Then ν is a *cluster point* of

- heteroclinic points belonging to x^* and y^*,
- homoclinic points belonging to x^* and homoclinic points belonging to y^*,
- periodic points,

whose orbits *all* run in $U(\Lambda)$. The proof of this statement is left to the reader (one constructs suitable ε-pseudo orbits and then applies the shadowing lemma).

We are going to demonstrate that the *complexity* of the orbit structure near a homoclinic orbit can be described *statistically* by the embedding of *Bernoulli systems*. Thus, we shall obtain orbits that are determined by *random sequences*. Introducing the *finite alphabet*

$$A = \{1, 2, \ldots, a\}, \quad a \geq 2,$$

the *space of the two-sided sequences of symbols* from the alphabet is the metric space

$$\Sigma_A = \{s = (s_j)_{j\in\mathbb{Z}} \mid s_j \in A\}$$

equipped with the metric

$$d(s,t) = \sum_{j\in\mathbb{Z}} \frac{1}{2^{|j|}} \frac{|s_j - t_j|}{1 + |s_j - t_j|}, \quad s,t \in \Sigma_A.$$

The metric has the following *significance.* Two sequences $s,t \in \Sigma_A$ are close, if they agree over a long, central string: $s_j = t_j$ for all $|j| \le N$ with N large. More precisely, the following lemma applies.

Lemma III.13. *If* $s,t \in \Sigma_A$, *then*

(i) $d(s,t) < \frac{1}{2^{N+1}} \Longrightarrow s_j = t_j, |j| \le N,$

(ii) $s_j = t_j, \quad |j| \le N \Longrightarrow d(s,t) \le \frac{1}{2^{N-1}}.$

Proof. (i) Assume $s_j \neq t_j$ for an integer $|j| \le N$, then $\frac{|s_j - t_j|}{1+|s_j-t_j|} \ge \frac{1}{2}$ and therefore

$$d(s,t) \ge \frac{1}{2^{|j|}} \frac{|s_j - t_j|}{1 + |s_j - t_j|} \ge \frac{1}{2^{|j|+1}} \ge \frac{1}{2^{N+1}}.$$

(ii) If $s_j = t_j$ for $|j| \le N$ then, due to $\frac{|s_j - t_j|}{1+|s_j-t_j|} \le 1$,

$$d(s,t) = \sum_{|j|>N} \frac{1}{2^{|j|}} \frac{|s_j - t_j|}{1 + |s_j - t_j|} \le 2 \sum_{j=N+1}^{\infty} \frac{1}{2^j} = \frac{1}{2^{N-1}},$$

as claimed in the lemma □

Lemma III.14 (Properties of (Σ_A, d)). *The metric space* (Σ_A, d) *is* ***compact*** *and* ***perfect****, i.e., every point is a cluster point.*

Proof [Finiteness of the alphabet]. If $(s^n)_{n\ge1}$ is a sequence in Σ_A, hence $s^n = (s_j^n)_{j\in\mathbb{Z}}$, we shall construct a convergent subsequence in the metric space (Σ_A, d). Since the alphabet is *finite*, there exists for every index $j \in \mathbb{Z}$ a symbol $a_j \in A$ such that $s_j^n = a_j$ for infinitely many n. We now choose a subsequence of s^n whose sequences all have the value a_0 at the index 0. From this subsequence, we choose again a subsequence whose sequences have the values a_{-1} and a_1 at the indices -1 and 1. Iterating the procedure, we finally arrive at a subsequence which, in view of Lemma III.13, converges to the sequence $a := (a_j)_{j\in\mathbb{Z}}$ in Σ_A. Hence, Σ_A is compact.

If $s \in \Sigma_A$ we choose a symbol $a \in A$ such that $s \neq (\dots, a, a, a, \dots)$ and define the sequence s^n by $s_j^n = s_j$ for $|j| \le n$ and $s_n^j = a$ for $|j| > n$. Then $s^n \to s$ and (s^n) is not constant. Therefore, Σ_A is perfect. □

Definition. If (M, d) is a metric space, the mapping $\varphi\colon M \to M$ is called *expansive*, if there exists a universal constant $\alpha > 0$ such that for all $x \neq y$ in M there exists an integer $N \geq 0$ for which $d(\varphi^N(x), \varphi^N(y)) \geq \alpha$.

In case that φ is bijective, one merely requires the existence of an integer $N \in \mathbb{Z}$ having the above property.

If the mapping is expansive, the iterates of two different points visibly separate from each other in the course of time, regardless of how close to each other they start. The dynamical system (M, φ) therefore shows a *sensitive dependence* on the initial conditions.

Equivalently, the map is expansive if there exists a constant $\alpha > 0$, having the property

$$d(\varphi^j(x), \varphi^j(y)) < \alpha \text{ for all } j \implies x = y.$$

Hence, if two orbits stay close for all times, then they must be identical.

Proposition III.15. *The dynamical system (Λ, φ) on a* hyperbolic *set Λ of a diffeomorphism φ is expansive.*

Proof [Shadowing lemma]. We assume that δ_0 and $\varepsilon_0 = \varepsilon_0(\delta_0)$ are as in the shadowing lemma and let $p = (p_j)_{j \in \mathbb{Z}}$ and $q = (q_j)_{j \in \mathbb{Z}}$ be two orbits on Λ satisfying

$$|p_j - q_j| = d(\varphi^j(p_0), \varphi^j(q_0)) \leq \delta_0, \quad j \in \mathbb{Z}.$$

Then q is an ε-pseudo orbit (with $\varepsilon = 0$) which is shadowed by the orbit p. Since the shadowing orbit is *unique* and since q is also an orbit, we conclude that $p = q$. □

Definition. The *shift map* σ on the space Σ_A is the mapping $\sigma\colon \Sigma_A \to \Sigma_A$, defined by

$$s \mapsto \sigma(s) = (\sigma(s)_j)_{j \in \mathbb{Z}}, \quad \sigma(s)_j := s_{j+1}.$$

The dynamical system (Σ_A, σ) is called a *Bernoulli system*.

Lemma III.16 (Properties of (Σ_A, σ)). *The Bernoulli system (Σ_A, σ) has the following properties.*

(i) *$\sigma\colon \Sigma_A \to \Sigma_A$ is a homeomorphism.*

(ii) *There exists a countable and **dense** set of **periodic** points of σ. All periods exist.*

(iii) *The system is **transitive**.*

(iv) *The system is **expansive**.*

(v) *If s, t are two periodic points, then the set of points $r \in \Sigma_A$ satisfying $\sigma^j(r) \to \mathcal{O}(s)$ as $j \to +\infty$ and $\sigma^j(r) \to \mathcal{O}(t)$ as $j \to -\infty$ is **dense** in Σ_A. These points are therefore heteroclinic to the orbits $\mathcal{O}(s)$ and $\mathcal{O}(t)$.*

Proof. (i) The bijectivity is obvious, the continuity of σ (and hence of σ^{-1}) immediately follows from the definition of the metric.

(ii) The periodic points are precisely the periodic sequences in Σ_A. Obviously, the set of such sequences is countable and dense.

(iii) We construct a dense orbit by taking a symbol sequence which sticks together *all* the possible symbol sequences of finite length $1, 2, 3, \dots$. According to Lemma III.13, the iterates of this sequence come arbitrarily close to every element of Σ_A proving that the system is transitive.

(iv) We define for $b \in A$ the set

$$\Sigma_A^b = \{s \in \Sigma_A \mid s_0 = b\}.$$

Then

$$\alpha := \inf_{b \neq c} d(\Sigma_A^b, \Sigma_A^c) \geq \frac{1}{2}.$$

If $s \neq t$, then $s_j \neq t_j$ for a some integer $j \in \mathbb{Z}$. Due to $\sigma^j(s)_0 = s_j$, and similarly for t, we have

$$\sigma^j(s) \in \Sigma_A^{s_j}, \quad \text{and} \quad \sigma^j(t) \in \Sigma_A^{t_j},$$

so that $d(\sigma^j(s), \sigma^j(t)) \geq d(\Sigma_A^{s_j}, \Sigma_A^{t_j}) \geq \alpha$.

(v) We assume that s, t are periodic points, then the sequences (s_j), (t_j) are periodic. Let m and n be the periods of the sequences (s_j) and (t_j), and set

$$S := (s_0, \dots, s_m), \quad T := (t_0, \dots, t_n),$$

then every symbol sequence

$$r = (\dots, T, T, X, S, S, \dots)$$

having an arbitrarily chosen finite central block X converges under the iterates of the shift map to $\mathcal{O}(s)$ and under the iterates of the inverse shift map to $\mathcal{O}(t)$. According to Lemma III.13 these sequences are dense in Σ_A. ☐

We come to the central result of this chapter.

Theorem III.17 (S. Smale). *Let ν be a transversal homoclinic point of the diffeomorphism φ belonging to the hyperbolic fixed point 0. Let U be an open neighborhood of the closure of the homoclinic orbit $\Lambda = \overline{\mathcal{O}(\nu)} = \mathcal{O}(\nu) \cup \mathcal{O}(0)$ and let A be a finite alphabet. Then there exists an integer $K \geq 1$ and a homeomorphism*

$$\psi \colon \Sigma_A \to \psi(\Sigma_A) =: M \subset U$$

having the following properties.

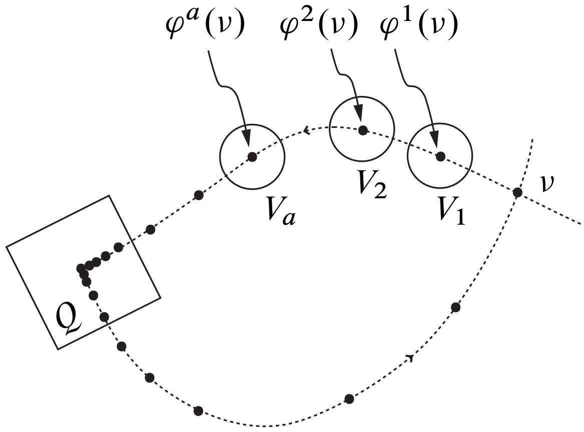


Figure III.9. The geometric construction in the proof of Theorem III.17.

(i) *The compact set M is invariant under φ^K, so that $\varphi^K(M) = M$.*
(ii) *For every point $m \in M$ the orbit $\mathcal{O}(m) \subset U$ under φ lies in the open set U.*
(iii) *$\psi \circ \sigma = \varphi^K \circ \psi$, so that the following diagram is commutative.*

$$\begin{array}{ccc} M & \xrightarrow{\varphi^K} & M \\ \uparrow{\scriptstyle\psi} & & \uparrow{\scriptstyle\psi} \\ \Sigma_A & \xrightarrow{\sigma} & \Sigma_A \end{array}$$

The theorem guarantees a *continuous embedding* of the Bernoulli system (Σ_A, σ) into $\mathbb{R}^n$ as a *subsystem* of the dynamical system $(\mathbb{R}^n, \varphi^K)$.

Proof [Proposition III.5, Theorem III.7]. Since Σ_A is compact, every injective and continuous mapping $\psi : \Sigma_A \to \psi(\Sigma_A) \subset V$ is a homeomorphism onto its image.

(1) *Strategy of the geometric construction.* If $A = \{1, 2, \dots, a\}$ is the alphabet, we choose *open neighborhoods* V_j of the finitely many homoclinic points $\varphi^j(\nu)$ for $1 \le j \le a$, satisfying

$$\overline{V_i} \cap \overline{V_j} = \emptyset, \quad i \neq j,$$

and set $V = \bigcup_{1 \le j \le a} V_j$. We shall construct for every sequence $s = (s_j)_{j \in \mathbb{Z}} \in \Sigma_A$ a point $p_0 = \psi(s) \in V$ having the following property. If $\Phi := \varphi^K$ is the iterated map for a suitable integer $K \ge 1$, then,

$$\Phi^j(p_0) \in V_{s_j} \quad \text{for all } j \in \mathbb{Z}.$$

Thus for *every random sequence* $s \in \Sigma_A$ there exists an orbit $(\Phi^j(p_0))_{j \in \mathbb{Z}}$ of Φ satisfying $\Phi^j(p_0) \in V_{s_j}$. Hence this orbit visits all *randomly* chosen sets V_b where $b \in A$.

(2) *Construction of the ε-pseudo orbit.* We choose $\delta > 0$ so small that the δ neighborhood of Λ is contained in U and the δ neighborhood of the homoclinic point $\varphi^j(\nu)$ is contained in V_j for every $j \in A$. Let $\varepsilon > 0$ be the real number ε which corresponds to δ in the shadowing lemma. Since ν is *a homoclinic point*, there *exists* an ε neighborhood Q of 0 and an integer $N \geq 1$ such that

$$\varphi^{s_j+N-1}(\nu) \in Q \quad \text{and} \quad \varphi^{s_j-N+1}(\nu) \in Q \quad \text{for all } s_j \in A.$$

For $s \in \Sigma_A$ the ε-pseudo orbit $q(s) = (q(s)_j)_{j\in\mathbb{Z}} \subset \Lambda$ is constructed in the following way. If $s = (\dots, s_{-1}, s_0, s_1, \dots) \in \Sigma_A$ is given, the points $q(s)_j$ are defined by identifying the points of the first scheme below with the corresponding points of the second scheme

$$\begin{array}{cccccccc}
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
q_{-N+1} & \cdots & q_{-1} & q_0 & q_1 & \cdots & q_{N-1} & q_N \\
q_{N+1} & \cdots & q_{2N-1} & q_{2N} & q_{2N+1} & \cdots & q_{3N-1} & q_{3N} \\
q_{3N+1} & \cdots & q_{4N-1} & q_{4N} & q_{4N+1} & \cdots & q_{5N-1} & q_{5N} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots
\end{array}$$

and

$$\begin{array}{cccccccc}
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\varphi^{s_0-N+1} & \cdots & \varphi^{s_0-1} & \varphi^{s_0} & \varphi^{s_0+1} & \cdots & \varphi^{s_0+N-1} & 0 \\
\varphi^{s_1-N+1} & \cdots & \varphi^{s_1-1} & \varphi^{s_1} & \varphi^{s_1+1} & \cdots & \varphi^{s_1+N-1} & 0 \\
\varphi^{s_2-N+1} & \cdots & \varphi^{s_2-1} & \varphi^{s_2} & \varphi^{s_2+1} & \cdots & \varphi^{s_2+N-1} & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots
\end{array}$$

where, to save space we have abbreviated $q_j = q(s)_j$ and $\varphi^j = \varphi^j(\nu)$. According to our construction the sequence $q(s)$ is indeed an ε-pseudo orbit on the hyperbolic set Λ. Moreover, it has the following crucial properties,

$$\begin{aligned} q(s)_{j+2N} &= q(\sigma(s))_j, & j \in \mathbb{Z}, \\ q(s)_{j2N} &= \varphi^{s_j}(\nu) \in V_{s_j}, & j \in \mathbb{Z}. \end{aligned}$$

(3) *Definition of the map $\psi \colon \Sigma_A \to \mathbb{R}^n$.* In view of the shadowing lemma, there exists an orbit $p(s)$ in the neighborhood U which is a δ shadowing orbit of the ε-pseudo orbit $q(s)$, so that $p(s)_j = \varphi^j(p(s)_0)$ and $|p(s)_j - q(s)_j| \leq \delta$ for all $j \in \mathbb{Z}$. From $q(s)_0 = \varphi^{s_0}(\nu) \in V_{s_0}$, we conclude that the point $p(s)_0$ lies in V_{s_0}.

We now define the mapping ψ by setting $\psi(s) = p(s)_0$ for $s \in \Sigma_A$ and introduce the diffeomorphism $\Phi := \varphi^{2N}$ (hence $K = 2N$). We claim that

$$\Phi^j(\psi(s)) \in V_{s_j}, \quad j \in \mathbb{Z}.$$

Indeed, due to our construction, $|p(s)_{j2N} - q(s)_{j2N}| \leq \delta$ and we conclude from $q(s)_{j2N} = \varphi^{s_j}(\nu) \in V_{s_j}$ that the point $p(s)_{j2N}$ lies in V_{s_j}, so that

$$p(s)_{j2N} = \varphi^{j2N}(p(s)_0) = \Phi^j(\psi(s)) \in V_{s_j}, \quad j \in \mathbb{Z}.$$

(4) *In order to prove the equation* $\psi \circ \sigma = \Phi \circ \psi$ we first fix $s \in \Sigma_A$ and abbreviate $p_0 := p(s)_0 = \psi(s)$, so that $|\varphi^j(p_0) - q(s)_j| \leq \delta$ for all $j \in \mathbb{Z}$. In particular, $|\varphi^{j+2N}(p_0) - q(s)_{j+2N}| \leq \delta$ and recalling $q(s)_{j+2N} = q(\sigma(s))_j$ and $\Phi = \varphi^{2N}$, we obtain the estimates

$$|\varphi^j(\Phi(p_0)) - q(\sigma(s))_j| \leq \delta, \quad j \in \mathbb{Z}.$$

Hence, $(\varphi^j(\Phi(p_0)))_{j\in\mathbb{Z}}$ is a δ-shadowing orbit of the ε-pseudo orbit $q(\sigma(s))$. According to the definition of the map ψ we have

$$|\varphi^j(\psi(\sigma(s))) - q(\sigma(s))_j| \leq \delta, \quad j \in \mathbb{Z},$$

so that $(\varphi^j(\psi(\sigma(s))))_{j\in\mathbb{Z}}$ is also a δ-shadowing orbit of the pseudo orbit $q(\sigma(s))$. From the uniqueness of the δ-shadowing orbit we conclude that

$$\varphi^j(\Phi(p_0)) = \varphi^j(\psi(\sigma(s))), \quad j \in \mathbb{Z}.$$

In particular, setting $j = 0$ and recalling $p_0 = \psi(s)$ we obtain the equation

$$\Phi(\psi(s)) = \psi(\sigma(s)).$$

This equation holds true for every $s \in \Sigma_A$, as we wanted to prove.

(5) *In order to verify the injectivity of the map* ψ, we take two elements $s \neq s'$ in Σ_A and set $p_0 := \psi(s)$ and $p'_0 := \psi(s')$, so that, according to our construction,

$$\Phi^j(p_0) \in V_{s_j}, \quad \Phi^j(p'_0) \in V_{s'_j}.$$

There exists an integer $j \in \mathbb{Z}$ for which $s_j \neq s'_j$ and hence $V_{s_j} \cap V_{s'_j} = \emptyset$. Consequently, $\Phi^j(p_0) \neq \Phi^j(p'_0)$ and since the map Φ can be inverted, we conclude $p_0 \neq p'_0$, and so $\psi(s) \neq \psi(s')$.

(6) In order to show that the mapping $\psi\colon \Sigma_A \to V$ *is continuous* we take the convergent sequence $s^{(n)}$ in Σ_A satisfying $s^{(n)} \to s$ in (Σ_A, d) and show that $\psi(s^{(n)}) \to \psi(s)$ in $\mathbb{R}^n$. Arguing by contradiction we find a subsequence $\psi(s^{(n)})$ such that

$$|\psi(s^{(n)}) - \psi(s)| \geq \varepsilon^* > 0$$

for all n. Since $\overline{V}$ is bounded, there exists a convergent subsequence again denoted by $\psi(s^{(n)})$. Denote its limit by ξ, then $\xi \in \overline{V}$ and $|\xi - \psi(s)| \geq \varepsilon^*$. We claim that $(\varphi^j(\xi))_{j\in\mathbb{Z}}$ is a δ-shadowing orbit of the pseudo orbit $q(s)$. In view of the uniqueness of the shadowing orbit, it then follows that $\xi = \psi(s)$ which is the desired contradiction. Fixing j we obtain by the triangle inequality

$$\begin{aligned} |\varphi^j(\xi) - q(s)_j| \le |\varphi^j(\xi) - \varphi^j(\psi(s^{(n)}))| \\ + |\varphi^j(\psi(s^{(n)})) - q(s^{(n)})_j| \\ + |q(s^{(n)})_j - q(s)_j|. \end{aligned}$$

We estimate the terms on the right-hand side. The first term converges to 0 as $n \to \infty$. The second term is estimated by $\le \delta$, as we have already seen in (4) (where $s^{(n)} = s$ and $\psi(s^{(n)}) = p_0$). The third term vanishes for n large enough, in view of the convergence $s^{(n)} \to s$ in the special metric of the Bernoulli system. All in all, $|\varphi^j(\xi) - q(s)_j| \le \delta$ for every $j \in \mathbb{Z}$, as claimed.

(7) The statement (i) of the theorem follows from $\Phi(m) = \psi \circ \sigma \circ \psi^{-1}(m) \in \operatorname{Im}\psi$ if $m \in \operatorname{Im}\psi$. Finally, the points $\varphi^j(\psi(s))$ which, by construction, belong to the δ-shadowing orbit of an ε-pseudo orbit on the set Λ, necessarily lie in a δ neighborhood of Λ and so the statement (ii) holds true by construction. The proof of Theorem III.17 is complete. □

By means of the homeomorphism $\psi\colon \Sigma_A \to \psi(\Sigma_A) =: M \subset \mathbb{R}^n$ the dynamical system $(M, \Phi) := (\psi(\Sigma_A), \varphi^K)$ inherits the properties of the Bernoulli system (Σ_A, σ) listed in Lemma III.16.

Corollary III.18. *The subsystem $(M, \Phi) := (\psi(\Sigma_A), \varphi^K)$ introduced in Theorem III.17 has the following properties.*

(i) *Φ is transitive on M.*

(ii) *The periodic points of the map Φ are countable and dense in M.*

(iii) *Φ is an expansive map on M.*

(iv) *The heteroclinic points of Φ (to periodic orbits) are dense in M.*

Proof. We only have to verify the expansiveness. Introducing the positive number

$$\alpha := \min_{b \neq c \in A} d(\overline{V_b}, \overline{V_c}) > 0,$$

we shall show for $x \neq y$ in M that there exists an integer $j \in \mathbb{Z}$ such that $|\Phi^j(x) - \Phi^j(y)| \ge \alpha$. To do so, we take s, s' in Σ_A satisfying $x = \psi(s)$ and $y = \psi(s')$. Then $s_j \neq s'_j$ for some integer $j \in \mathbb{Z}$. According to the geometric construction of Theorem III.17 we know that that $\Phi^j(x) \in V_{s_j}$ and $\Phi^j(y) \in V_{s'_j}$ and so, $|\Phi^j(x) - \Phi^j(y)| \ge \alpha$ as claimed. □

III.4 Existence of transversal homoclinic points

We illustrate the chaotic behavior caused by a homoclinic point with the example of a periodically perturbed mathematical pendulum. Starting with the unperturbed situation and assuming all physical constants to be normalized, the mathematical pendulum is determined by the differential equation

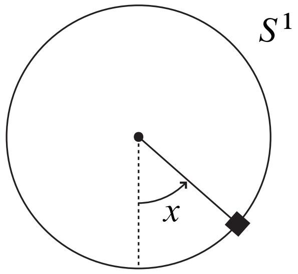


Figure III.10. The pendulum.

$$\ddot{x} + \sin x = 0$$

of second order, where $x \pmod{2\pi}$ is the angle of the swing of the pendulum. Written as an equivalent system of differential equations of first order, the pendulum is described by

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\frac{dV}{dx}(x), \quad V(x) := -\cos x. \end{cases}$$

Due to the periodicity, the phase space is equal to $S^1 \times \mathbb{R}$, it is, however, more convenient to work in the *covering space* $\mathbb{R}^2$. We write the system as a vector field in $\mathbb{R}^2$,

$$\dot{z} = X(z) \in \mathbb{R}^2, \quad z = (x, y) \in \mathbb{R}^2.$$

The flow $\varphi^t(z)$ of the vector field X is defined by the unique solutions of the Cauchy initial value problem

$$\begin{cases} \frac{d}{dt}\varphi^t(z) = X(\varphi^t(z)), \quad t \in \mathbb{R}, \\ \varphi^0(z) = z. \end{cases}$$

For fixed z the curve $t \mapsto \varphi^t(z) \in \mathbb{R}^2$ is the solution of the initial value problem having the initial conditions z at the time $t = 0$. For fixed time t the mapping $z \mapsto \varphi^t(z)$ is a diffeomorphism of $\mathbb{R}^2$.

The orbits are quickly sketched, since there exists an *integral* $H : \mathbb{R}^2 \to \mathbb{R}$ given by

$$H(x, y) = \frac{1}{2}y^2 + V(x) = \frac{1}{2}y^2 - \cos x.$$

We recall that an integral of the vector field X is a function $H : \mathbb{R}^2 \to \mathbb{R}$ satisfying $dH(X) = 0$. Equivalently, the flow of X leaves the function H invariant, so that $H(\varphi^t(z)) = H(z)$ for all t and z. Therefore, the orbits lie on the level lines

$$E_c := \{(x, y) \in \mathbb{R}^2 \mid H(x, y) = c\},$$

consisting of the two branches $y = \pm\sqrt{2(c + \cos x)}$. Figure III.11 shows that the mathematical pendulum possesses the following *orbit types*.

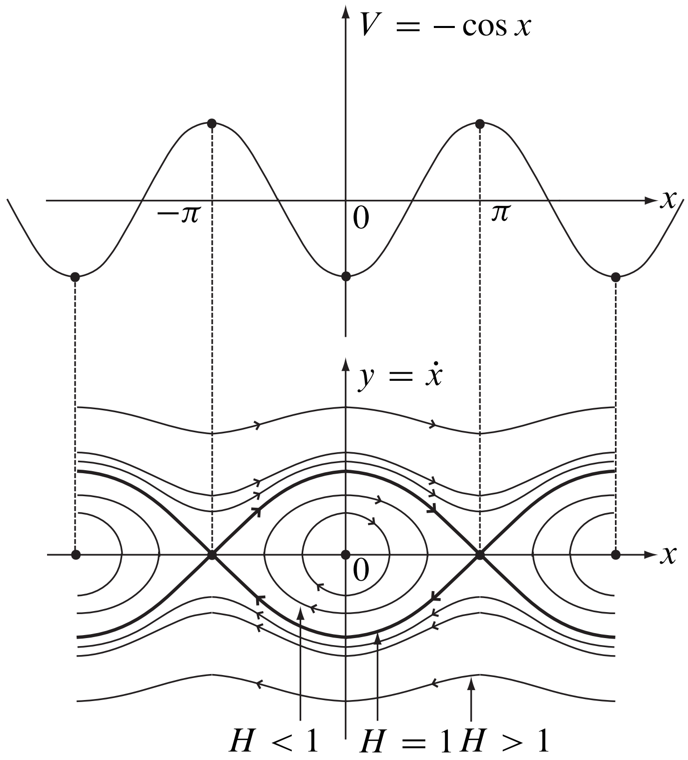
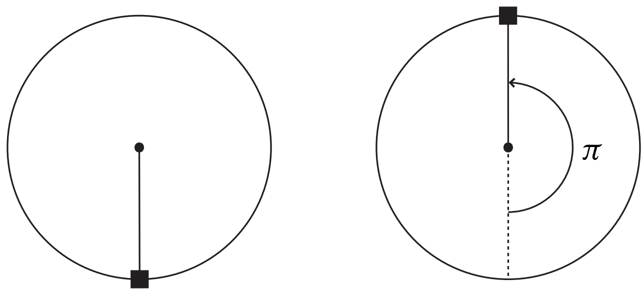
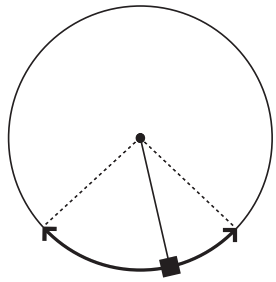


Figure III.11. Level sets of the integral H. The separatrix is marked.

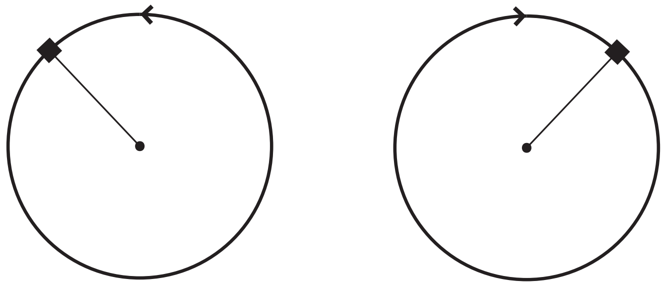
Equilibrium points. The equilibrium points are, on one hand, the constant orbits in the level set $\{H = -1\}$, these are the so-called elliptic equilibrium points located in $(x, y) = (2\pi n, 0)$ (on the left figure). On the other hand, the hyperbolic equilibrium points located in $(x, y) = ((2n + 1)\pi, 0)$ are on the level set $\{H = 1\}$.



Oscillation. The oscillations around the lowest point are on the level sets $\{-1 < H < 1\}$, in Figure III.12 described by the closed curves.



Rotations. The rotational solutions lie on the level sets $\{H > 1\}$. The angle is either strictly increasing (left) or else strictly decreasing (right).



Heteroclinic orbits. The level set $\{H = 1\}$ carries the homoclinic orbits (in $S^1 \times \mathbb{R}$) and the heteroclinic orbits (in $\mathbb{R}^2$) respectively, and the hyperbolic equilibrium points. This level set is called a *separatrix* because it separates the oscillations from the rotations.

Keeping the time $T > 0$ fixed, the flow in time T,

$$\varphi^T : \mathbb{R}^2 \to \mathbb{R}^2,$$

is a diffeomorphism possessing the hyperbolic fixed points $P_n = ((2n+1)\pi, 0)$ for $n \in \mathbb{Z}$. This is easily verified using Lemma II.12. They are 2π-periodically distributed (in the projection on $S^1 \times \mathbb{R}$ they all correspond to the same point). Their stable and unstable invariant manifolds coincide in the sense that $W_+(P_n) = W_-(P_{n+1})$ for all $n \in \mathbb{Z}$. We now *perturb* the pendulum by means of a *time T-periodic excitation* and consider the equation

$$\ddot{x} + \sin x = \mu \sin \omega t, \quad T = \frac{2\pi}{\omega}.$$

The energy function H is *no longer* an integral of the system and the orbit structure changes drastically. The new vector field

$$\dot{z} = X(t, \mu, z) \in \mathbb{R}^2$$

is now *time dependent* and *T-periodic* in time t, so that $X(t+T,\mu,z) = X(t,\mu,z)$ for all t,μ,z. The flow solves the initial value problem

$$\begin{cases} \frac{d}{dt}\varphi^t(z,\mu) = X(t,\mu,\varphi^t(z,\mu)), & t \in \mathbb{R}, \\ \varphi^0(z,\mu) = z. \end{cases}$$

Due to the *uniqueness* of the Cauchy initial value problem, it follows from the 2π periodicity of the vector field X in the variable x that

$$\varphi^t(z + 2\pi j e_1, \mu) = \varphi^t(z,\mu) + 2\pi j e_1,$$

for all $t,\mu \in \mathbb{R}$ and $j \in \mathbb{Z}$, where $e_1 = (1,0)$. Moreover, it follows from the T-periodicity of the vector field in time t that

$$\varphi^{t+T}(z,\mu) = \varphi^t(\varphi^T(z,\mu))$$

for every $t \in \mathbb{R}$ and $z \in \mathbb{R}^2$ (recall that the relation $\varphi^t \circ \varphi^s = \varphi^{t+s}$ is only valid for the flow of a time independent vector field). Keeping the parameter μ fixed, the mapping

$$\psi(z) := \varphi^T(z,\mu) \colon \mathbb{R}^2 \to \mathbb{R}^2$$

is a diffeomorphism satisfying $\psi^j(z) = \varphi^{jT}(z,\mu)$ for every $z \in \mathbb{R}^2$.

Let us assume that there exists a solution $x(t)$ of the equation $\ddot{x} + \sin x = \mu \sin \omega t$ possessing *infinitely many zeros* (mod 2π), at the times $(t_k)_{k\in\mathbb{Z}}$ which are all nondegenerate. The times are ordered according to $t_k < t_l$ if $k < l$, so that

$$x(t_k) = 0 \mod 2\pi, \quad \dot{x}(t_k) \neq 0, \quad k \in \mathbb{Z}.$$

In other words, the pendulum passes the lowest point infinitely often with a non-vanishing velocity. We associate with this solution a two-sided sequence $\sigma(x(t)) = (\sigma_k(x(t)))_{k\in\mathbb{Z}}$, defined by

$$\sigma_k(x(t)) = \operatorname{sign}(\dot{x}(t_k)) = \begin{cases} +1, & \dot{x}(t_k) > 0, \\ -1, & \dot{x}(t_k) < 0. \end{cases}$$

In the *unperturbed* case $\mu = 0$ there exist precisely three types of such sequences, namely

(a) constant $+1$, i.e., $\sigma_k(x(t)) = +1$ for all $k \in \mathbb{Z}$,
(b) constant -1, i.e., $\sigma_k(x(t)) = -1$ for all $k \in \mathbb{Z}$,
(c) alternating, i.e., $\sigma(x(t)) = (\ldots,+1,-1,+1,-1,+1,\ldots)$.

In sharp contrast to this unperturbed situation, the perturbed mathematical pendulum possesses a solution for every prescribed random sequence as the following theorem shows.

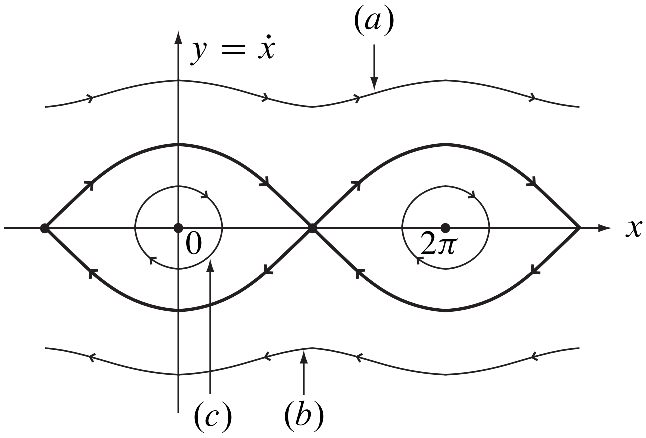


Figure III.12. Types in the unperturbed case.

Theorem III.19. *Let $U \subset \mathbb{R}^2$ be an open neighborhood of the separatrix (in the case $\mu = 0$). If $|\mu| > 0$ is sufficiently small, then there exists for every two-sided sequence $(s_k)_{k\in\mathbb{Z}}$ of integers $s_k \in \{-1, 1\}$ a solution $x(t)$ of the perturbed pendulum equation $\ddot{x} + \sin x = \mu \sin \omega t$ such that $(x(t), \dot{x}(t)) \in U$ which possesses infinitely many nondegenerate zeros (mod 2π) satisfying*

$$\sigma_k(x(t)) = s_k, \quad k \in \mathbb{Z}.$$

In addition, for every finite sequence $s_k \in \{-1, 1\}$ where $-N \leq k \leq M$ there exists a solution $x(t)$ possessing only finitely many nondegenerate zeros (mod 2π) and solving the equations $\sigma_k(x(t)) = s_k$ for $-N \leq k \leq M$. The same applies to half finite sequences s_k for $-\infty < k \leq M$ or for $-N \leq k < \infty$.

In short, one can prescribe any sequence of directions with which the pendulum should consecutively pass through the lowest point and there exists a solution doing precisely that.

Proof [Transversal heteroclinic point, shadowing lemma]. Assuming $\mu \neq 0$, we consider the diffeomorphism ψ of $\mathbb{R}^2$, defined by the time T flow map

$$\psi(z, \mu) := \psi_\mu(z) := \varphi^T(z, \mu),$$

at the time $T = 2\pi/\omega > 0$. In the case $\mu = 0$, the map ψ has the hyperbolic fixed point $P := P_{-1} = (-\pi, 0)$. We shall show that also the diffeomorphism ψ_μ has a *unique hyperbolic fixed point $P(\mu)$ near $P = P(0)$*, which depends differentiably on μ, if μ is small enough. For this, we define the mapping $F\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ by

$$F(z, \mu) = \varphi^T(z, \mu) - z.$$

If $\mu = 0$ then $F(P,0) = \phi^T(P,0) - P = 0$ and the partial derivative in the variable z is given by

$$D_1(P,0) = D_1\varphi^T(P,0) - \mathbb{1} \in \mathcal{L}(\mathbb{R}^2).$$

The linear mapping $D_1 F(P,0)$ is an isomorphism, since the hyperbolic matrix $D_1\varphi^T(P,0)$ does not have an eigenvalue equal to 1. In a neighborhood of $\mu = 0$ there exists, by the implicit function theorem, a unique continuously differentiable function $\mu \mapsto P(\mu) \in \mathbb{R}^2$, solving $F(P(\mu),\mu) = 0$ and $P(0) = P$. In other words,

$$P(\mu) = \psi_\mu(P(\mu))$$

is a fixed point of the mapping ψ_μ. The eigenvalues of the derivative $d\psi_\mu(P(\mu))$ depend continuously on μ, hence the linear map $d\psi_\mu(P(\mu))$ possesses for small μ an eigenvalue whose absolute value is > 1 and an eigenvalue whose absolute value is < 1. Consequently, $P(\mu)$ is a hyperbolic fixed point of ψ_μ, if μ is small. Also the points $P(\mu) + 2n\pi$ are hyperbolic fixed points and $P(\mu) + 2n\pi = P_{-1}(\mu) + 2n\pi = P_{n-1}(\mu)$.

From the proof of Theorem II.8 (construction of h) we know that the *local invariant manifolds* issuing from the hyperbolic fixed point $P(\mu)$, denoted by

$$W^{\pm}_{\mathrm{loc}}(P(\mu)),$$

depend differentiably on μ (by the implicit function theorem). For small μ they can, therefore, be represented locally as graphs over the invariant manifolds of the unperturbed system (the branches of the separatrix). If $t \mapsto \gamma(t)$ is a *heteroclinic solution* in the unperturbed case $\mu = 0$ having the x-coordinate at time $t = 0$ equal to $(\gamma(0))_1 = 0$, then $\gamma(t)$ lies on the separatrix. In formulas,

$$\frac{d}{dt}\gamma(t) = X(0,\gamma(t)), \quad t \in \mathbb{R},$$

and $\gamma(t) \to P = P_{-1}$ as $t \to -\infty$ and $\gamma(t) \to P + 2\pi e_1 = P_0$ as $t \to +\infty$.

Denoting by $n(\gamma(t))$ the unit normal vector of the homoclinic orbit γ in the point $\gamma(t)$ as depicted in Figure III.14, we can represent the relevant pieces of the invariant manifolds as follows:

$$\{\gamma(r) + u^-(r,\mu)\cdot n(\gamma(r)) \mid -\infty < r \le M\} \subset W_-(P_{-1}(\mu))$$

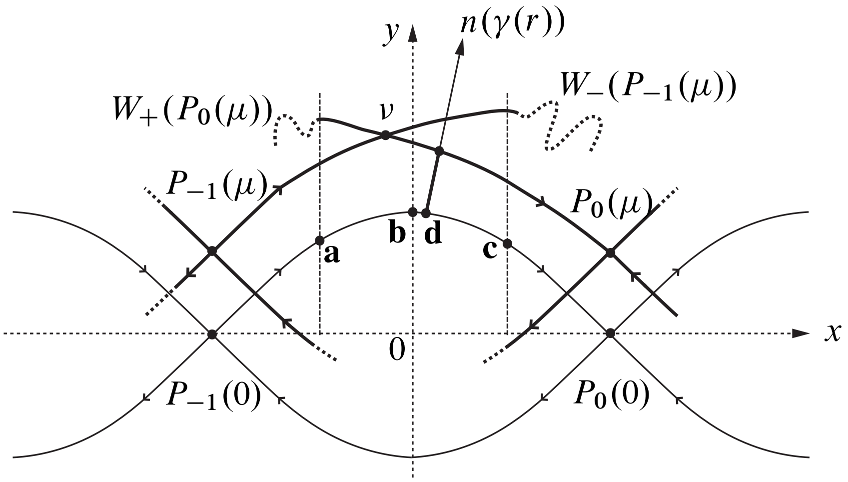
and

$$\{\gamma(r) + u^+(r,\mu)\cdot n(\gamma(r)) \mid -M \le r < \infty\} \subset W_+(P_0(\mu)),$$

with a sufficiently large constant $M > 0$ and where in the case $\mu = 0$ the functions $u^+(r,0) = u^-(r,0) = 0$ vanish.

If for a parameter value $r \in \mathbb{R}$,

$$u^-(r,\mu) = u^+(r,\mu) \quad \text{and} \quad \frac{\partial}{\partial r}u^-(r,\mu) \ne \frac{\partial}{\partial r}u^+(r,\mu),$$



Legend: **a**: $\gamma(-M)$ **b**: $\gamma(0)$ **c**: $\gamma(M)$ **d**: $\gamma(r)$

Figure III.13. The perturbed invariant manifolds possessing a transversal intersection.

then we have found the *transversal* intersection point

$$\nu := \gamma(r) + u^-(r, \mu) \cdot n(\gamma(r)) \in W_-(P_{-1}(\mu)) \cap W_+(P_0(\mu)).$$

In order to study the first-order term in μ of the function $(u^- - u^+)$ we introduce the so-called Melnikov function

$$d(r) := \frac{\partial}{\partial \mu}(u^- - u^+)|_{\mu=0}(r).$$

If $d(r_0) = 0$ and $\frac{d}{dr} d(r_0) \neq 0$, then there exists a transversal intersection point near $\gamma(r_0)$, for small $\mu \neq 0$. This follows from

$$(u^- - u^+)(r, \mu) = \mu\big(d(r) + O(\mu)\big)$$

in view of the implicit function theorem. The first approximation $d(r)$ can be explicitly calculated by means of the following *Melnikov formula*.

Theorem III.20 (Melnikov). *Let*

$$\dot{z} = f(z) + \mu g(t, z) \in \mathbb{R}^2,$$

$z \in \mathbb{R}^2$ *be a smooth vector field satisfying* div $f = 0$, *where* g *is a* T*-periodic vector field for some* $T > 0$, *so that* $g(t + T, z) = g(t, z)$. *We assume that for* $\mu = 0$ *there exists a homoclinic* (*resp. heteroclinic*) *orbit* γ *of the vector field* f, *hence satisfying* $\frac{d}{dt}\gamma(t) = f(\gamma(t))$ *for all* $t \in \mathbb{R}$ *and*

$$\gamma(t) \to P, \quad t \to -\infty,$$
$$\gamma(t) \to Q, \quad t \to +\infty$$

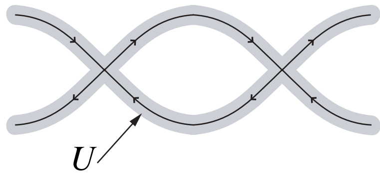


Figure III.14. A neighborhood U of the separatrix.

for two hyperbolic fixed points P and Q of f. Then, setting $f = (f_1, f_2)$ and $g = (g_1, g_2)$ the following formula holds true:

$$d(r) = \frac{1}{|f(\gamma(r))|} \int_{-\infty}^{\infty} (f_1 g_2 - f_2 g_1)(s, \gamma(r+s))\, ds.$$

For a *proof* we refer to C. Robinson in [91, S. 304]. In order to apply the formula to our pendulum, we consider the upper branch of the unperturbed separatrix,

$$\begin{cases} -\pi \le x \le \pi, \\ y = +\sqrt{2(1+\cos x)} = 2\cos(\frac{x}{2}) \end{cases}$$

where $y = \dot{x}$. The solution of the equation $\dot{x} = 2\cos(\frac{x}{2})$ is given by

$$x(t) = 2\arcsin(\tanh(t)), \quad t \in \mathbb{R},$$

and differentiating we obtain

$$\dot{x}(t) = y(t) = \frac{2}{\cosh(t)}.$$

Hence $\gamma(t) = (x(t), y(t))$ is the heteroclinic orbit. Inserting the curve γ into the Melnikov formula results in

$$d(r) = \frac{1}{|X(\gamma(r))|} \frac{2\pi \sin(\omega r)}{\cosh(\frac{\omega\pi}{2})}.$$

The function $d(r)$ has the *nondegenerate* zeros $r = \frac{\pi}{\omega} j$ for all $j \in \mathbb{Z}$. Therefore, there exists a transversal heteroclinic point ν. In the same way, there exists near the lower branch of the separatrix a transversal heteroclinic point η. The closure of the heteroclinic orbits is the *hyperbolic* set

$$\Lambda = \bigcup_{j,k\in\mathbb{Z}} \psi^j(\nu + 2\pi k e_1) \cup \{P_k(\mu)\} \cup \psi^j(\eta + 2\pi k e_1).$$

In order to finish the proof of Theorem III.19 we choose a neighborhood U of the separatrix and we choose the parameters ε, δ as in the shadowing lemma. For the given sequence $s = (s_k)_{k\in\mathbb{Z}}$ we construct the following ε-pseudo orbit q, described by Figure III.15. If $s_0 = 1$, we start in the heteroclinic point $q_0 = \nu$ and if $s_0 = -1$ we start in the heteroclinic point $q_0 = \eta$. Then we follow the heteroclinic orbit $\psi^j(\nu)$, resp. $\psi^j(\eta)$ into the $(\varepsilon/2)$-neighborhood of the next hyperbolic fixed point. There one has again two possibilities. If $s_1 = 1$ we jump onto the heteroclinic orbit of the upper branch to the *right* while if $s_1 = -1$ we jump onto the heteroclinic orbit of the lower branch to the *left*, and so on.

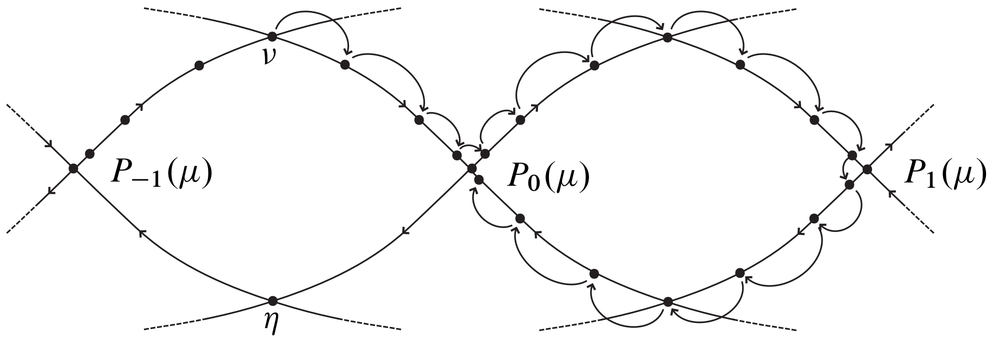


Figure III.15. The ε-pseudo orbit associated with the sequence $(s_k) = (\dots, s_0, s_1, s_2, \dots) = (\dots, 1, 1, -1, \dots)$.

The associated δ-shadowing orbit $p = (p_j)_{j\in\mathbb{Z}}$ guaranteed by the shadowing lemma,

$$p_j = \psi^j(p_0) = \varphi^{jT}(p_0, \mu), \quad j \in \mathbb{Z},$$

lies on the desired *solution* $t \mapsto \varphi^t(p_0, \mu)$ of the perturbed vector field $X(t, \mu, z)$ starting at the point $\varphi^0(p_0, \mu) = p_0$ at the time $t = 0$ and remaining in the open neighborhood U of the separatrix. Of course, this solution loses a lot of time near the hyperbolic equilibrium points, away from these neighborhoods it moves quite fast. We point out that all the solutions found this way start in a small neighborhood of the homoclinic points ν resp. η!

The passages near the transversal heteroclinic points ν resp. η correspond to the passages of the pendulum through the point $x = 0 \mod 2\pi$, which is the lowest position of the pendulum. In order to obtain a solution defined by a finite sequence (s_k), one constructs an ε-pseudo orbit q as before which, however, at the ends is equal to the orbits of hyperbolic fixed points. Then, the corresponding solution of the pendulum equation makes finitely many swings back and forth and then remains almost immobile near the highest position of the pendulum! This completes the proof of Theorem III.19. $\square$

For a detailed study of the chaotic behavior of the periodically perturbed pendulum we refer to U. Kirchgraber and D. Stoffer in [59].